THE UNIVERSITY OF ALBERTA

ELASTIC STABILITY OF FRAMES

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES

IN PARTIAL FULFILMENT OF THE REQUIREMENTS

FOR THE DEGREE OF MASTER OF SCIENCE

DEPARTMENT OF CIVIL ENGINEERING

EDMONTON, ALBERTA

NOVEMBER, 1964

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the faculty of Graduate Studies for acceptance, a thesis entitled

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ABSTRACT

This thesis is concerned with elastic critical loads of single story, single bay, rectangular rigid frames subjected to uniform vertical load and concentrated horizontal load. These frames are hence subjected to axial loads and primary bending moments. The frames investigated have hinged bases, fixed bases, or base fixity intermediate between these two extremes. Slope-deflection equations, modified for the effects of axial loads, are used for the structural analyses of the frames and an iterative technique involving the use of an electronic digital computer is employed to solve the governing equations for each frame.

Critical loads are defined as the highest load on a load-deflection plot for the frame. The results indicate that the fixed-base frame will sustain an elastic critical load about four times greater than the corresponding hinged-base frame, for a constant value of horizontal load. When horizontal load is present, both frames exhibit large sway deflections before the critical load is reached. For both frames, the elastic critical load decreases with increasing horizontal load, the frame dimensions and ratio of member rigidities remaining constant. The elastic critical load also decreases as the length of beam increases with respect to the length of columns, the horizontal load and ratio of member rigidities remaining constant. Furthermore, the elastic critical load increases as the beam rigidity increases with respect to the column rigidity, the horizontal load and frame dimensions remaining constant. The presence of even a small amount of base restraint increases the critical load appreciably above that of a hinged base frame.

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This amount of base restraint is available under presently used column bases that are commonly assumed in design to be hinged bases.

This method of solution may be readily adapted to stability analysis of other practical building frames subjected to axial loads and primary bending moments.

ACKNOWLEDGEMENTS

The author wishes to express his sincere appreciation

to Prof. J. Longworth, Department of Civil Engineering, University of Alberta, for his suggestions, criticisms, and encouragement during the preparation of this thesis;

to staff members of the Department of Civil Engineering, University of Alberta, for their time and donation of reference material;

to staff members of the University of Alberta Computing Centre for their helpful suggestions and running of the computer programs.

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NOMENCLATURE

C = Non-dimensional carry-over factor
E = Modulus of elasticity of frame material
EI = flexural rigidity of frame member

$(EI/L)_c$ = Flexural stiffness of column members
$(EI/L)_B$ = Flexural stiffness of beam
$(EI/L)_{RB}$ = Flexural stiffness of restraining beam
FEM = Fixed end moment
I = Moment of inertia of cross-section of frame member

K = Stiffness factor of frame member = SEI/L
$k = \sqrt{P/EI}$
L = Length of frame member

L_c - Length of column members
L_B - Length of beam
L_{RB} - Length of restraining beam

M_{ij} = Moment at end i of member ij

P = Axial load

$P_E = \pi^2 EI/L^2$ = Simple Euler load of frame member
$(P_E)_c$ = Simple Euler load of column members
$(P_E)_B$ = Simple Euler load of beam

$RP = \beta/\alpha$ = Ratio of horizontal to vertical load

$REI = EI_B/EI_c$

S = Non-dimensional stiffness coefficient

w = Uniform lateral load
x = Coordinate direction
y = Coordinate direction

θ = Sway rotation of frame member
θ_i = Rotation of joint i

α = Total vertical load$/(P_E)_c = wL_B/(P_E)_c$
β = Total horizontal load$/(P_E)_c = RP\alpha$

$\lambda = L_B/L_c$

$\gamma = (EI/L)_c/(EI/L)_B$

$\eta = (EI/L)_{RB}/(EI/L)_B = EI_{RB}/EI_B$

Δ = Sway deflection

$\rho = P/P_E$

$\phi = kL = \sqrt{PL^2/EI} = \pi\sqrt{P/P_E} = \pi\sqrt{\rho}$

CHAPTER I

INTRODUCTION

1.1 STRUCTURAL DESIGN CRITERIA

The use of limit design in present-day engineering practice requires the satisfying of three main criteria for structural damage. One of these is the ultimate or plastic strength criterion, which is satisfied by insuring that the maximum load on a structure times a safety factor does not exceed the ultimate or plastic strength of the structure. A second criterion is that the buckling strength must not be exceeded, which has led to the concept of design for stability. Buckling occurs in the elastic or inelastic ranges when the structure passes from one deformation configuration into another without a change in the load. A third criteria to be satisfied is that the deformation of the structure must not exceed specified limits, in order that the serviceability of the structure is not impaired.

The usual design procedure is to proportion the members on the basis of their ultimate strength, and then perform separate checks to insure satisfaction of the stability and deflection criteria. The individual members and the structure as a unit must satisfy all criteria.

1.2 CONCEPT OF FRAME INSTABILITY

Structural design for stability requires consideration of three general types of instability, these being:

(i) Local buckling of elements of the cross-section of individual members.

(ii) Instability of individual members under the action of axial force and bending moment. This category includes instability of beam-columns due to excessive bending, and lateral-torsional buckling.

(iii) Overall instability of the entire structure, often referred to as frame instability. This arises from unfavorable deflections in the frame as a unit, the mutual interaction of all members being the important feature.

The first two types of instability have been extensively studied for many years, and satisfactory design recommendations have evolved from this work. The subject of frame instability has become more prominent with the advent of plastic design. Limitations on the application of the plastic theory have arisen from the difficulty of dealing with instability problems in partially plastic structures. Furthermore, modern architectural requirements have resulted in the use of frames virtually unbraced by walls or partitions, with light floors, and non-structural fireproofing. Without the built-in safety factors of conventional construction, frame instability becomes an important consideration.

Although frame instability is sometimes of importance even in the elastic range, it is frequently ignored in design. The complexity of the problem may be the main reason for the comparatively small amount of attention devoted to it, and this complexity increases greatly when plastic deformation occurs.

Frame instability can be illustrated as in Figure 1-1 for a portal frame which is not prevented from sidesway. Two types of failure may occur:

(A) When a symmetrical frame is loaded by symmetrical vertical forces, it is possible that the frame may pass from a symmetrical stable deformation configuration to an unsymmetrical, unstable configuration (cases 1 and 2). At this instant, the total resistance to any lateral force or lateral move-

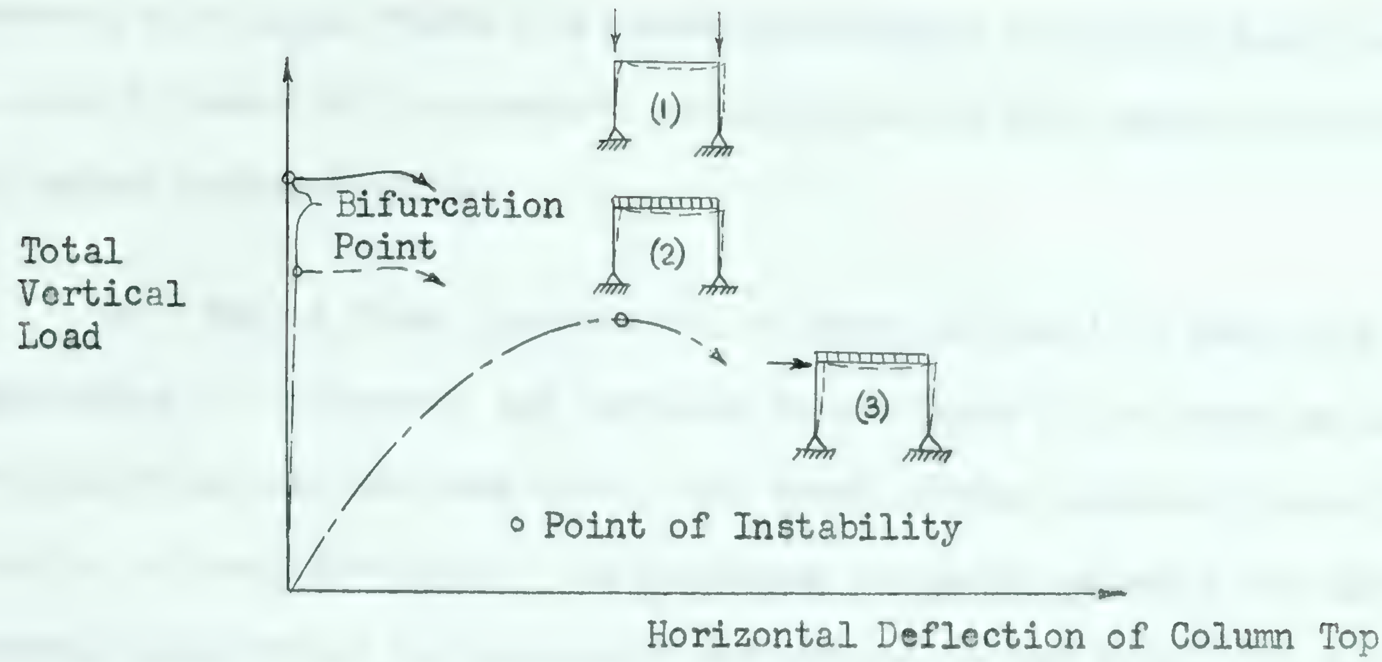


FIGURE 1-1

ment becomes zero. The load-deflection curve is characterized by a sudden shift from a situation where the applied load can increase with little or no deflection to one where large deflections develop without an increase of load. The behavior is analogous to that of a centrally loaded column, in which bifurcation of the equilibrium position is possible at a certain critical load. At a point of bifurcation, it is theoretically possible for either the stable or unstable configuration of the frame to be in equilibrium with the critical load.

It should be noted that if the frame is prevented from sidesway, the critical load would occur when the frame transforms to a symmetrical unstable configuration. This load is always higher than the sidesway critical load, and hence does not govern for frames free to sway.

There is a distinct difference in the nature of cases 1 and 2. In case 1, where the loads are applied directly to the columns, the frame theoretically carries no primary bending moment before buckling. Therefore, only the column action alone needs to be considered in the buckling analysis.

However, most rigid frames are primarily designed to support loads such as in case 2, where all the members are subjected to both axial force and bending moment before buckling.

(B) When a frame (symmetrical or unsymmetrical) is subjected to a combination of horizontal and vertical forces (case 3), or when an unsymmetrical frame carries beam loads, the frame deforms laterally upon the application of the first load. The resulting change in geometry may alter the carrying capacity of the individual columns, since the column top is no longer directly over the column base and hence additional bending moments are introduced by the vertical load. The whole structure becomes unstable in this deformed position much like an eccentrically loaded column. At the critical load, the structure continues to deform with no increase in load. Very little information is available concerning this type of failure.

The above discussion has inherently pointed out the three main causes of sway collapse. These are:

(i) vertical loads taken directly on columns.
(ii) vertical loads on beams causing primary bending.
(iii) horizontal loads (wind loads).

The first item (i) is the most important for tall building frames, since large axial forces cause a drastic reduction in the stiffness of the columns. The last two items are usually less important, although the exact extent of their influence has not been established.

The case of a frame subjected to a combination of horizontal and vertical loads as in case 3 represents a situation where all three causes of sway collapse are present. The critical load occurs when the horizontal deflection of the column tops increases with no increase in load. A linear elastic

analysis of the structural response of this frame does not exhibit such a maximum since actual collapse is due to factors not included in this type of analysis. These include primarily the effect of axial load on the stiffness of the frame members, the effect of plastic behavior, and the effect of finite displacements of the frame.

1.3 RELATION BETWEEN ELASTIC AND PLASTIC ANALYSES

In the calcuation of the elastic critical load, i.e. the critical load considering the frame to remain completely elastic, the effect of axial load on stiffness and the effect of finite displacements can be considered in the analysis. The calculation of the simple plastic collapse load by the usual simple plastic theory considers only the property of plasticity, although the effect of finite displacements can be considered by an extension of the simple plastic theory.[1] The relation of elastic and rigid plastic behavior to elastoplastic behavior can be most adequately described by reference to Figure 1-2.

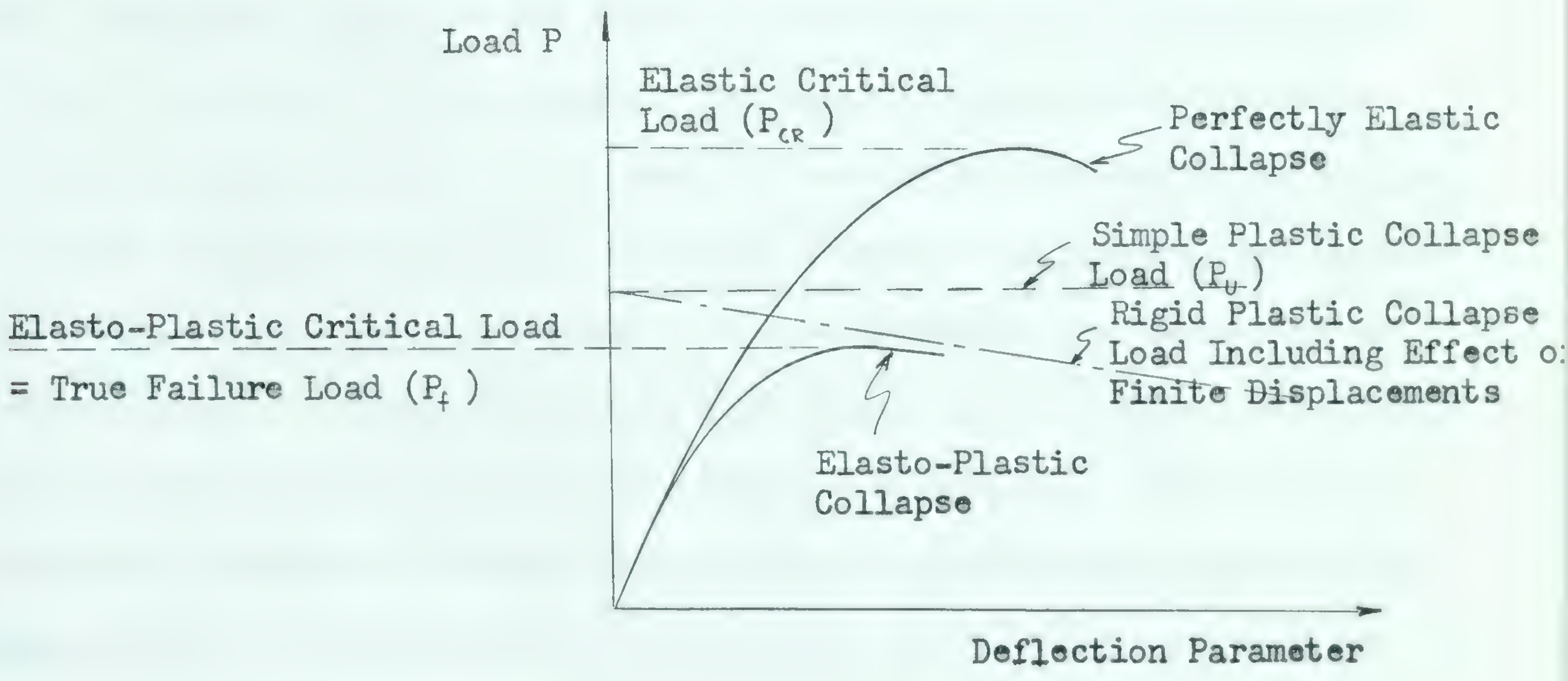


FIGURE 1-2

A perfectly elastic analysis considering the effect of axial load on stiffness and the effect of finite deformations produces a load deflection curve as shown. The elastic critical load P_{cr} then is the highest point on this load-deflection curve. The simple plastic theory yields the plastic collapse load (P_u) which is constant and independent of deflection. If the effect of finite displacements is included in the plastic analysis, a plastic collapse load is obtained which depends upon the deflection parameter. The true failure load is thus lower than that predicted by the simple plastic theory, and occurs due to instability of the elasto-plastic structure before a complete mechanism has been formed. The exact shape and position of the true (elasto-plastic) collapse curve is difficult to assess, but has been discussed by Horne.(1)

To obtain an insight into the reason for an instability collapse before a mechanism has been formed, Wood(2) has proposed the concept of a "deteriorated elastic structure". From an elastic stability point of view, the plastic zones in the structure are incapable of developing additional resistance to displacement. The deteriorated elastic structure at collapse can therefore be obtained by removing all yielded material from the frame and replacing plastic hinges by frictionless hinges. The elastic critical load of this deteriorated structure is called the "deteriorated critical load". A frame analyzed by Wood showed that the deteriorated critical load drops drastically as successive hinges are developed. Presumably, a reasonable estimate of the true failure load of the structure would be the "deteriorated critical load" of the structure for which all but the last hinge has formed.

The concept of the deteriorated critical load clarifies thinking on the subject of elastic-plastic instability of structures. However, it is of limited assistance only since the actual calculations are very difficult and time-consuming. For this reason, empirical approaches have been attempted to relate the true failure load (P_f) to the elastic critical load (P_{cr}) and the simple plastic collapse load (P_u). Merchant(3) has suggested a Rankine type formula as in Figure 1-3.

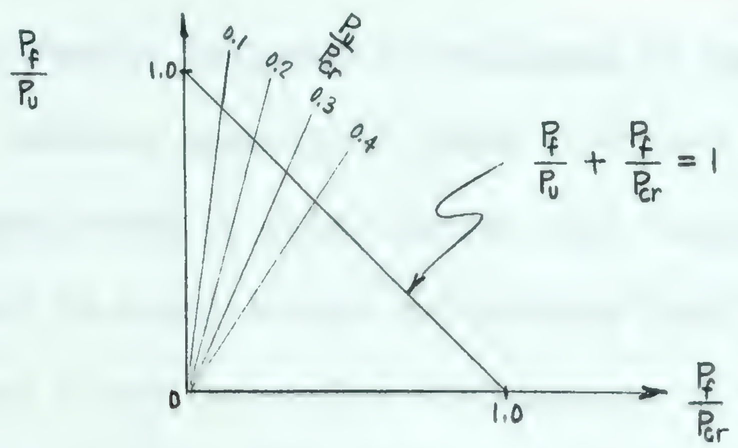


FIGURE 1-3

Experimental evidence to date has indicated that this formula is most successful when P_u/P_{cr} is small and the true collapse load is close to P_u. For larger values of P_u/P_{cr}, the formula is conservative. It is expected then that this formula is most applicable to low buildings, where in fact the elastic critical load is usually much higher than the collapse load predicted by the simple plastic theory. For taller buildings, the formula would provide a conservative estimate or lower limit to the true failure load.

1.4 LITERATURE REVIEW

Many investigations into the stability of frames have been carried out, almost all of this work being done under the assumption that the frame remains completely elastic i.e. elastic critical loads have been derived. In most cases, an approximate method of analysis has been used, whereby the actual loading on the structure is replaced by one consisting of forces

applied at the joints only. As long as no lateral loads are present, the members of the frame are then subjected to axial load only before buckling. This approximate method has the distinct advantage of reducing the amount of calculations required to determine the critical load, and is sufficiently accurate for many but not all cases. More recent investigators have used exact methods of analysis, whereby the effects of primary bending moments and associated deformations are included.

A recent bulletin(4) published by the Welding Research Council contains a tabular summary of known solutions for elastic critical loads for frames subjected to axial forces only (approximate method) and for frames subjected to axial forces and primary bending moment (exact method). In addition, a very extensive bibliography on the subject of frame stability is presented. Some of the more important of these solutions plus other investigations of interest are briefly discussed in the following two sections.

(A) Frames Carrying Axial Forces Only

Solutions for the elastic critical loads of frames using the approximate method of analysis have generally utilized some modified form of the slope-deflection equations or moment-distribution procedure, although solutions based on energy principles have been obtained. Since all members are subjected to axial force only, uniform yielding may be assumed for every section along each member; hence, an estimate of the inelastic buckling load can be obtained from these solutions by applying a tangent modulus modification. Lu(5) presents an excellent summary and brief discussion of these methods.

Bleich(6) and Timoshenko(7) include solutions for the elastic critical load of simple frames with either fixed or hinged column bases. Merchant

and others(8)(9)(10) obtained similar solutions for more complicated multistory, multibay structures, using type solution methods suggested by Bolton,(11) and stability functions developed at the University of Manchester by Livesly and Chandler.(12) Wood(2) calculated various "deteriorated critical loads" of a multistory frame as part of an investigation of his concept of a deteriorated elastic structure for stability analyses. Goldberg(13) investigated the critical loads of complete one-story building frames, where the roof is considered as an elastic restraint acting between the frames. This represents a more realistic approach to the actual situation. Galambos(14) demonstrated that only a small degree of restraint at the base of the columns increases the critical load considerably above that for hinged-base columns. A.S.C.E. Manual No. 41(15) contains graphically the solution to the elastic critical load of an analogous portal frame that was used to formulate a design rule for the new A.I.S.C. Code.

The theory of stability for structures whose members are subjected only to axial forces has been fully developed, and numerous methods for determining the critical load of various types of frameworks are now available. For complex structures, however, the numerical work involved in the analysis is often excessive, and simplifications in this field are required so that the required computations can be readily performed in a design office.

(B) Frames Carrying Axial Force and Bending Moment

Comparatively few solutions exist in the literature dealing with the elastic stability of frames subjected to loads which produce primary bending moments as well as axial forces in the members before buckling. The calculation of critical loads for frames using the available solutions is

often very tedious, and accurate numerical results have been obtained only for single bay, single story frames. Solutions to the same problems in the inelastic range are non-existent except for an initial attempt outlined in a thesis by Lu(5). Empirical approaches still remain as the only method of estimating a true elasto-plastic failure load for frames subjected to axial force and bending moment.

Bleich(6) gives a summary of the method followed by Chwalla in 1938, for a hinged rectangular frame subjected to two equal and symmetrically placed loads on the beam. Chwalla used the classical approach of integrating a system of differential equations and satisfying the associated boundary conditions. More recently, Masur and others(16) succeeded in extending the slope deflection and moment distribution methods to the analysis of the stability of frames carrying primary bending moment. Using similar means, Lu(5) obtained exact solutions for the elastic and inelastic critical loads of a hinged base rectangular frame subjected to a uniform load on the beam and two equal concentrated loads over the columns. This particular frame was chosen to simulate the lower story of a multi-story building.

Preliminary but unpublished work at the University of Alberta in 1963 has indicated that the existence of horizontal as well as vertical loads on a simple portal frame results in substantial sway deflections. This work thus suggests the use of a so-called "large deflection theory" to obtain more accurate elastic critical loads for this combination of loading.

The increasing use of electronic digital computers has served to further the use of existing methods in this field of endeavor. This thesis employs the use of such a computer in order to obtain elastic critical loads for building frames described in the next chapter.

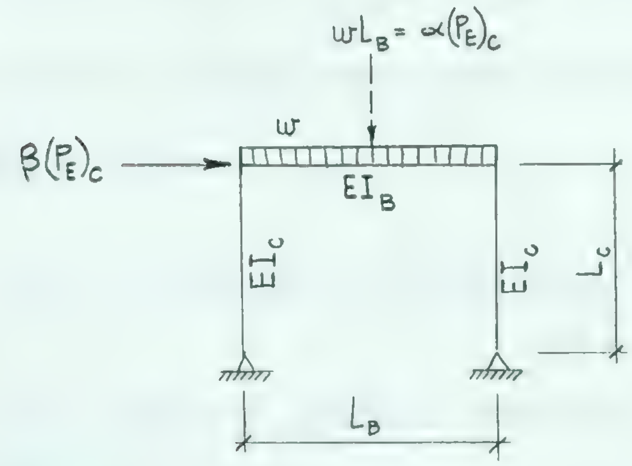
CHAPTER II

SCOPE OF PRESENT INVESTIGATION

Published results of investigations using approximate or exact methods of analysis cover cases where the frame is subjected to vertical load only. The investigation contained in this thesis was initiated to obtain exact solutions for the elastic critical loads of a simple rectangular frame under the action of a combination of vertical and horizontal loads. Also considered in this thesis is the effect of using a "large deflection theory," and the influence of partial base fixity on the elastic critical loads.

More specifically, the following frames, loadings, and ranges of variables were investigated:

(i) Frame with Pinned Base



$RP = \beta/\alpha$

$\lambda = L_B/L_C$

$REI = EI_B/EI_c$

FIGURE 2-1

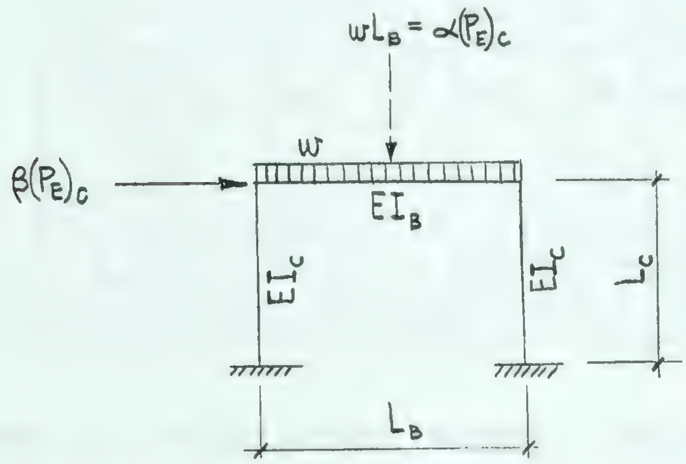
Small deflection and large deflection theories were investigated

for this frame and loading throughout the following ranges of variables: (S.D. - small deflection, L.D. - large deflection)

$\lambda = 1.0$			$\lambda = 2.0$			$\lambda = 3.0$			$\lambda = 4.0$		
REI	S.D.	L.D.	REI	S.D.	L.D.	REI	S.D.	L.D.	REI	S.D.	L.D.
1.0	✓	✓	2.0	✓	✓	3.0	✓	✓	4.0	✓	✓
2.0	✓		4.0	✓		6.0	✓		8.0	✓	
3.0	✓		6.0	✓		9.0	✓		12.0	✓	
4.0	✓	✓	8.0	✓	✓	12.0	✓	✓	16.0	✓	✓

For all combinations above, RP values of 0, 0.01, 0.04, 0.07, and 0.10 were used.

(ii) Frame With Fixed Base



$RP = \beta/\alpha$

$\lambda = L_B/L_c$

$REI = EI_B/EI_c$

FIGURE 2-2

Small deflection and large deflection theories were investigated for this frame and loading throughout the same ranges of variables as for the hinged-base frame.

(iii) Frame With Partial Base Fixity

The effect of partial restraint to rotation of the column bases on the elastic critical loads was simulated by inserting a restraining beam between the bases of the columns, as follows:

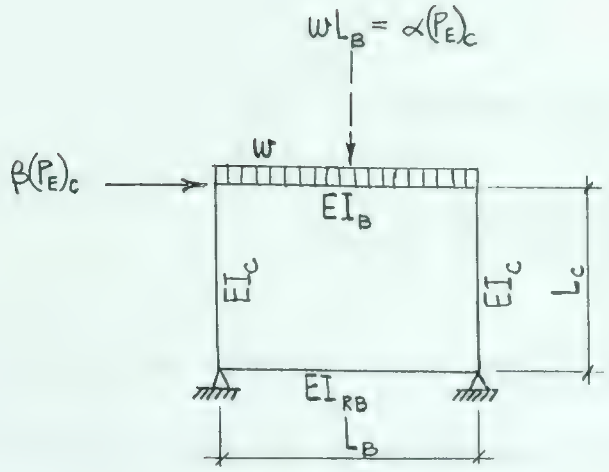


FIGURE 2-3

$RP = \beta/\alpha$

$\lambda = L_B/L_C$

$REI = EI_B/EI_C$

$\eta = EI_{RB}/EI_B$

The small deflection theory only was investigated throughout the following ranges of variables:

$\lambda = 1.0$		$\lambda = 2.0$		$\lambda = 3.0$		$\lambda = 4.0$	
REI	RP	REI	RP	REI	RP	REI	RP
1.0	0.0	2.0	0.0	3.0	0.0	4.0	0.0
1.0	0.10	2.0	0.10	3.0	0.1	4.0	0.1
4.0	0.0	8.0	0.0	12.0	0.0	16.0	0.0
4.0	0.10	8.0	0.10	12.0	0.1	16.0	0.1

For all combinations above, η values used to plot graphs were $\eta =$ 0.0, 0.20, 0.50, 1.0, 2.0, 4.0, and 8.0.

As can be seen, the large deflection theory for parts (i) and (ii) and the whole of part (iii) are investigated only at the extremes of the complete range of variables outlined in part (i). This approach was followed since an indication only of these results was considered necessary for practical use.

CHAPTER III

METHOD OF SOLUTION

In order to obtain exact solutions to the situations outlined, the effect of primary bending moments and associated deformations must be included. Since it is not possible with the techniques available to obtain unique mathematical solutions to problems of this kind, an iterative solution using the slope deflection equations modified to include the effect of axial load was chosen. The use of the University of Alberta's I.B.M. 1620 electronic digital computer made possible the use of a method such as this.

The necessary slope-deflection relationships may be developed by referring to Figure 3-1, in which a typical frame member connecting joints A and B is shown.

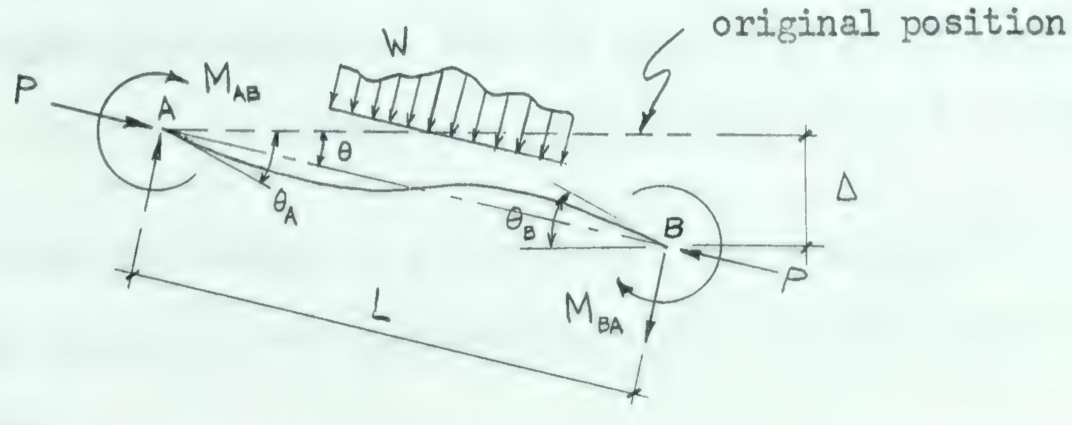


FIGURE 3-1

The typical member A-B is subjected to an axial force P, which is considered positive in compression, end moments M_{AB} and M_{BA}, end shears, and lateral load W between the ends. The joint rotations are θ_A and θ_B,

and the bar rotation is Θ, when the member is in the deflected position. End moments and rotations are defined as positive clockwise. The bending moments and rotations are defined as positive clockwise. The bending moments and joint rotations are related by the slope-deflection equations as follows:

$$M_{AB} = K_{AB}(\Theta_A + C_{AB}\Theta_B) - K_{AB}(1 + C_{AB}) \operatorname{Tan}\Theta \pm FEM$$

$$M_{BA} = K_{BA}(\Theta_B + C_{BA}\Theta_A) - K_{BA}(1 + C_{BA}) \operatorname{Tan}\Theta \pm FEM$$

In general, $M_{NF} = K_{NF}(\Theta_N + C_{NF}\Theta_F) - K_{NF}(1 + C_{NF}) \operatorname{Tan}\Theta \pm FEM$ where N denotes near end and F denotes far end. For relatively small rotations, Tan Θ effectively becomes Θ. In the slope deflection equations, K is the stiffness factor, C is the carry-over factor, and FEM is the fixed-end moment for the lateral load on the member. If the modulus of elasticity E is constant and the member is prismatic, then $K_{NF} = K_{FN} = K = \frac{SEI}{L}$

$$C_{NF} = C_{FN} = C$$

$$FEM_{NF} = -FEM_{FN}$$

The stiffness coefficient S and the carry-over factor C are functions of the axial load P in the member, as shown in Figures 3-2 and 3-3. These functions have been tabulated(12) for a wide range of axial loads, and are briefly developed in APPENDIX A. For the case of no axial load, the S and C functions reduce to the conventional values of 4.0 and 0.5 respectively.

The fixed end moment is a function of the magnitude of the axial load P and the magnitude and distribution of the lateral load W. For a prismatic member,

$$FEM_{NF} = -K(\Theta_{NS} + C\Theta_{BS})$$

where Θ_{NS} and Θ_{FS} are the rotations at the ends of the frame member when considered as a simply supported beam-column subjected to axial and lateral load. For a member subjected to a uniform lateral load w, and an axial

compressive force P, the expressions for the fixed end moment have been developed in APPENDIX B and are shown graphically in Figure 3-4. For the case of no axial load, the fixed end moment reduces to the conventional value of $0.08\bar{3}\, wL^2$.

The use of the slope deflection equations in the structural analysis of a building frame is illustrated fully in APPENDIX C. The slope-deflection equations are written for each joint of the frame, with the joint rotations and sway rotations as unknowns. By writing the equilibrium equation $\Sigma M = 0$ at hinged joints, it is possible to express these joint rotations in terms of others, thus reducing the number of unknowns. Writing the equilibrium equations $\Sigma M = 0$ at the remaining joints and $\Sigma F = 0$ for the frame as a whole, the governing equations for the solution of the joint and sway rotations of the frame are obtained. The equations will contain a load parameter of the frame which must be given a value in order to solve for the rotations. Conversely, if one of the rotations is specified in value, the load parameter and the remaining rotations can be solved. This approach is used in this thesis to solve for the elastic critical loads of the frames outlined in CHAPTER II.

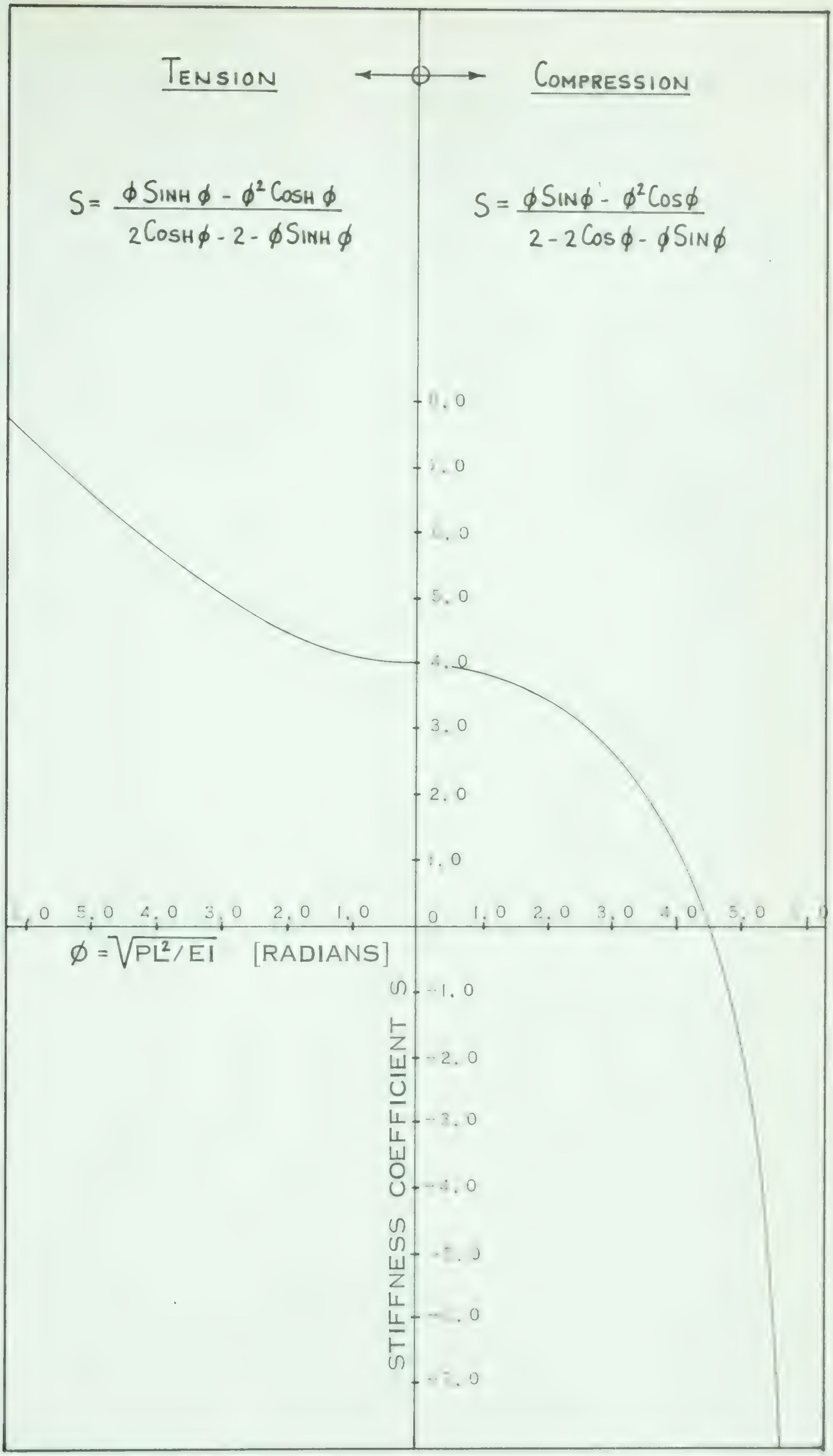


FIGURE 3 – 2 STIFFNESS COEFFICIENT FOR BENDING AND AXIAL LOAD

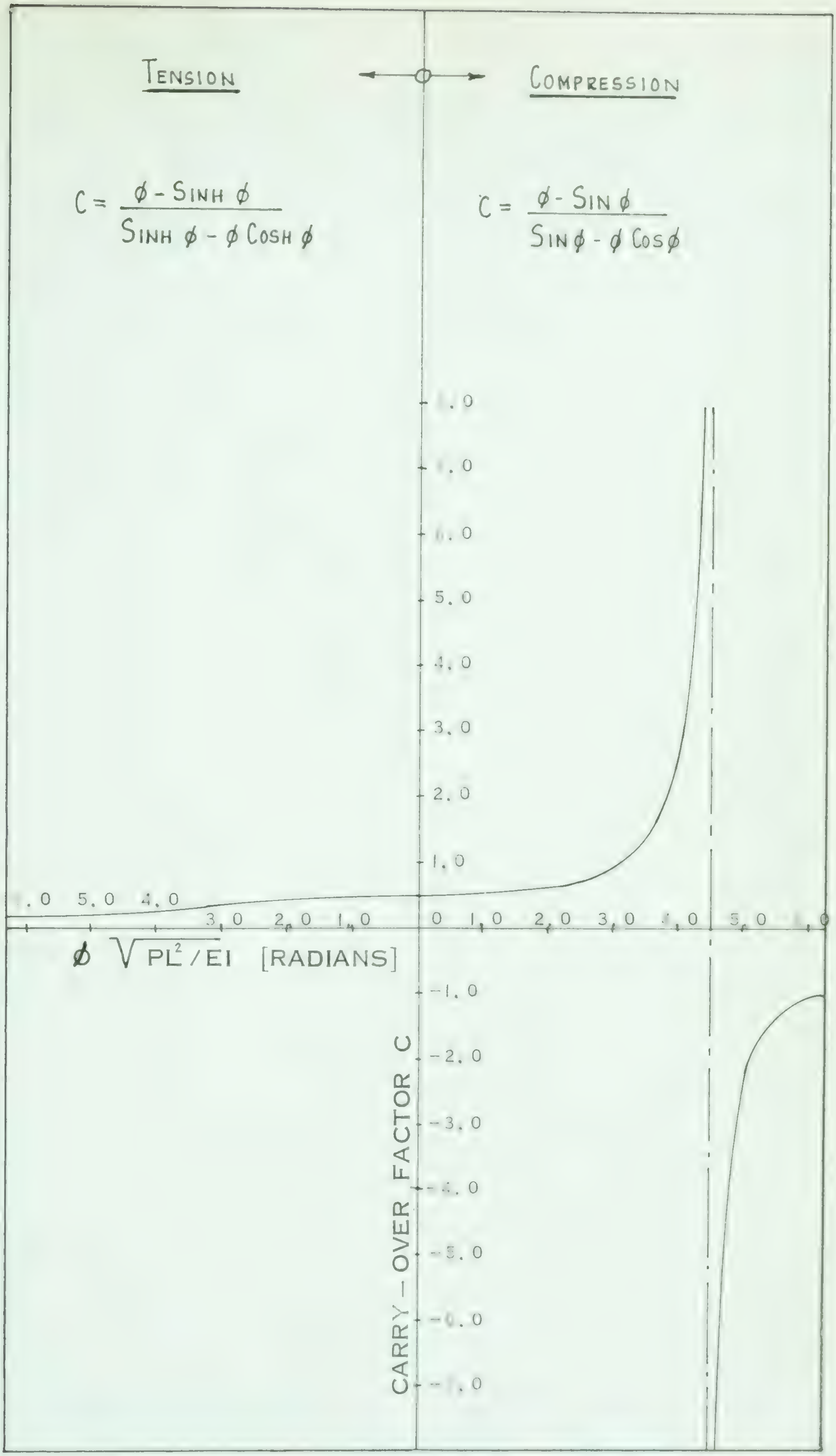


FIGURE 3 - 3 CARRY - OVER FACTOR FOR BENDING AND AXIAL LOAD

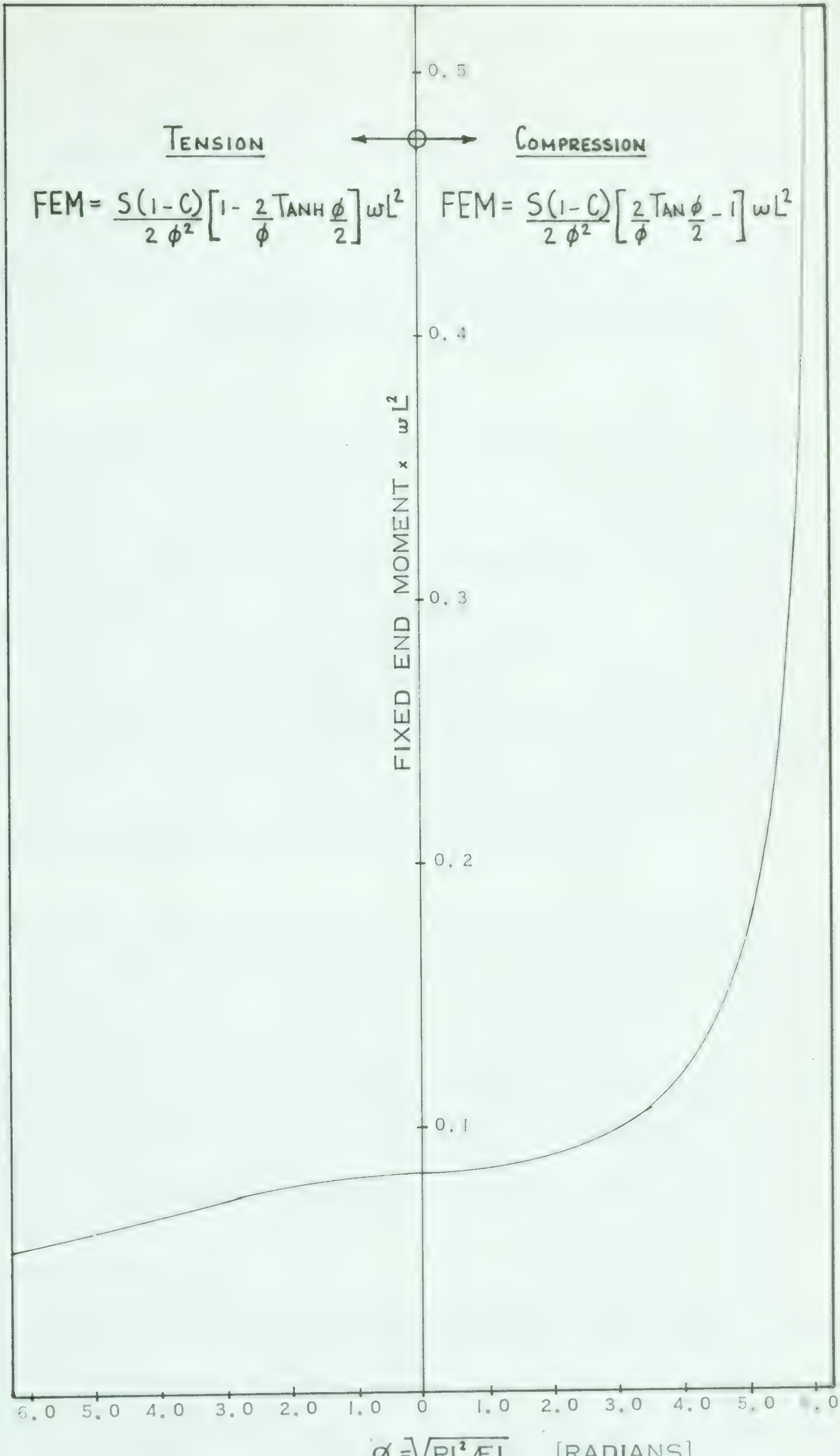


FIGURE 3-4 FIXED END MOMENT FOR UNIFORM LATERAL LOAD AND AXIAL LOAD

CHAPTER IV

PRESENTATION AND DISCUSSION OF RESULTS

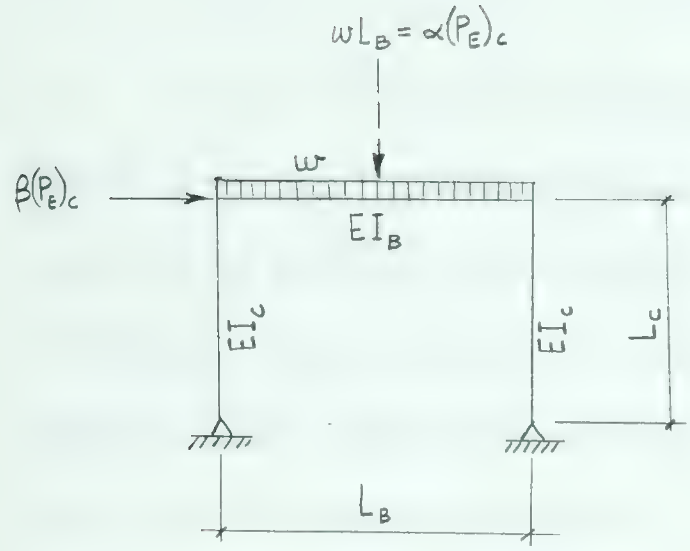
4.1 GENERAL

In present-day American design practice, it is quite common to assume column bases of frames as hinged. However, it has been shown(14) that presently used column bases offer a considerable restraint to base rotation. This restraint is enough to raise the critical load of some frames considerably above that of a hinged-base frame. Furthermore, the effect of cladding, composite action, infilling, and bracing in actual buildings is not considered in this or most other stability analyses, but is definitely beneficial towards preventing frame instability.

The results of this investigation are presented graphically on the following pages. For clarity, the results are divided into three separate sections. Sections 4-2 and 4-3 contain the results for the hinged-base frame and fixed-base frame respectively. Section 4-4 contains the results for the frame with partial base fixity, this case representing various degrees of base fixity between the extremes of hinged and fixed bases.

4-2 FRAME WITH HINGED BASES

The relationships between the load parameter α and sway rotation θ are shown in Figures 4-3 to 4-26 for the five values of horizontal load considered. The results of the "large deflection theory" are given following those of the small deflection theory.



$RP = \beta/\alpha$

$\lambda = L_B/L_C$

$REI = EI_B/EI_C$

FIGURE 4-1

For the case of no horizontal load, the maximum vertical load the frame can support if it remains elastic is $0.5\ (P_E)_c$, which occurs with an infinitely stiff beam member. This can be deduced by considering one of the column members in its buckled configuration under this condition, as in Figure 4-2.

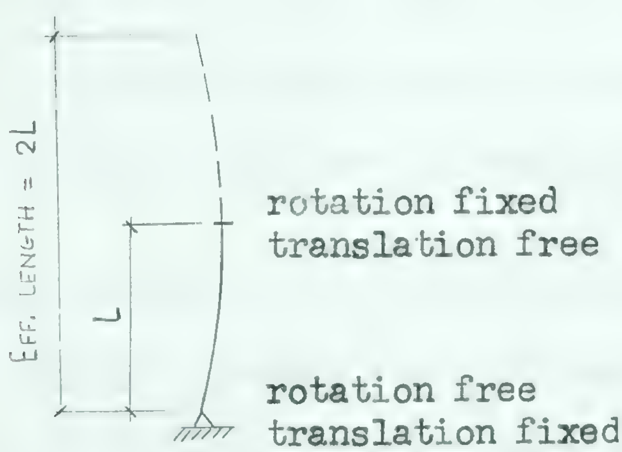


FIGURE 4-2

The effective length of this column member is 2L, making the elastic critical load equal to

$$\frac{\pi^2 EI}{(2L)^2} = 0.25\ \frac{\pi^2 EI}{L^2} = 0.25\ P_E$$

The frame has two such column supports, and hence its elastic critical

load becomes 2 x $0.25(P_E)_c = 0.50(P_E)_c$

It can be seen by considering the results of the small-deflection theory that the maximum vertical load sustained by this frame with no horizontal load present does in fact approach $0.5(P_E)_c$ as the beam stiffness increases. Conversely, when the beam stiffness is relatively small, the critical load drops considerably since the effective lengths of the columns are free to increase above 2L.

The existence of even a small horizontal load causes a significant increase in sway rotations, with larger sway rotations being evident for larger horizontal loads. The effect of horizontal loads on critical vertical loads was difficult to assess quantitatively, since a definite critical load was not always reached within a range of sway rotations less than 1.0. If it is assumed that the results are reliable throughout the range, and if the critical load is defined as the maximum load occurring in the range of $\theta \leq 1.0$, then it may be stated in general that the existence of a horizontal load does not decrease the critical load substantially, but has more effect on sway rotations.

The results of the "large-deflection theory" for only the extremes of the variables presented for the small-deflection theory are given in Figures 4-19 to 4-26. For all cases the total vertical load at a given sway rotation as predicted by the "large-deflection theory" is larger than that predicted by the small-deflection theory. This phenomenon is most pronounced at large sway rotations, while for very small sway rotations, the results are identical for both theories.

It should be noted that both the large and small deflection theories are based on the well-known approximate relationship $M = EI\ d^2y/dx^2$, which

describes the equilibrium configuration of a member, considering flexural deformations only. However, this equation is valid only if the difference between the total length along a member and its corresponding chord length is negligible. The exact value for curvature is $\frac{d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$, from which $M = \frac{EI\ d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$. If an error of 5% is allowed in this equation, then it can be shown that dy/dx must not exceed 10.4° or 0.1815 radians. An investigation of the data results from the computer programs has indicated that an error of this magnitude will occur for sway rotations of the frame exceeding $\theta \approx 0.20$ radians = 11.5°. Therefore, the results of both the small and "large" deflection theories must be considered inaccurate for sway rotations beyond $\theta = 0.20$ radians.

The above discussion indicates that a true large-deflection theory should be based on the exact relationship $M = \frac{EI\ d^2y/dx^2}{[1 + (dy/dx)^2]^{3/2}}$. The large-deflection theory used in this thesis must therefore be called a "pseudo large-deflection theory", since it considers only part of the difference between small and large deflections, as discussed in APPENDIX C.

It has been decided to treat the results of the small deflection theory as a reasonably conservative estimate of the elastic critical loads of this frame, and to use them in forming the following general observations:

(1) The elastic critical load decreases with increasing horizontal load, all other variables held constant. The percentage reduction from the case of no horizontal load is about the same for all combinations of λ and REI investigated.

(2) The elastic critical load decreases with increasing values of λ, the variables RP and REI being held constant.

(3) The elastic critical load increases with increasing values of REI, the variables RP and λ being held constant, i.e., the critical load increases with increasing beam stiffness.

A summary of the elastic critical loads obtained is given in tabular form following the graphs.

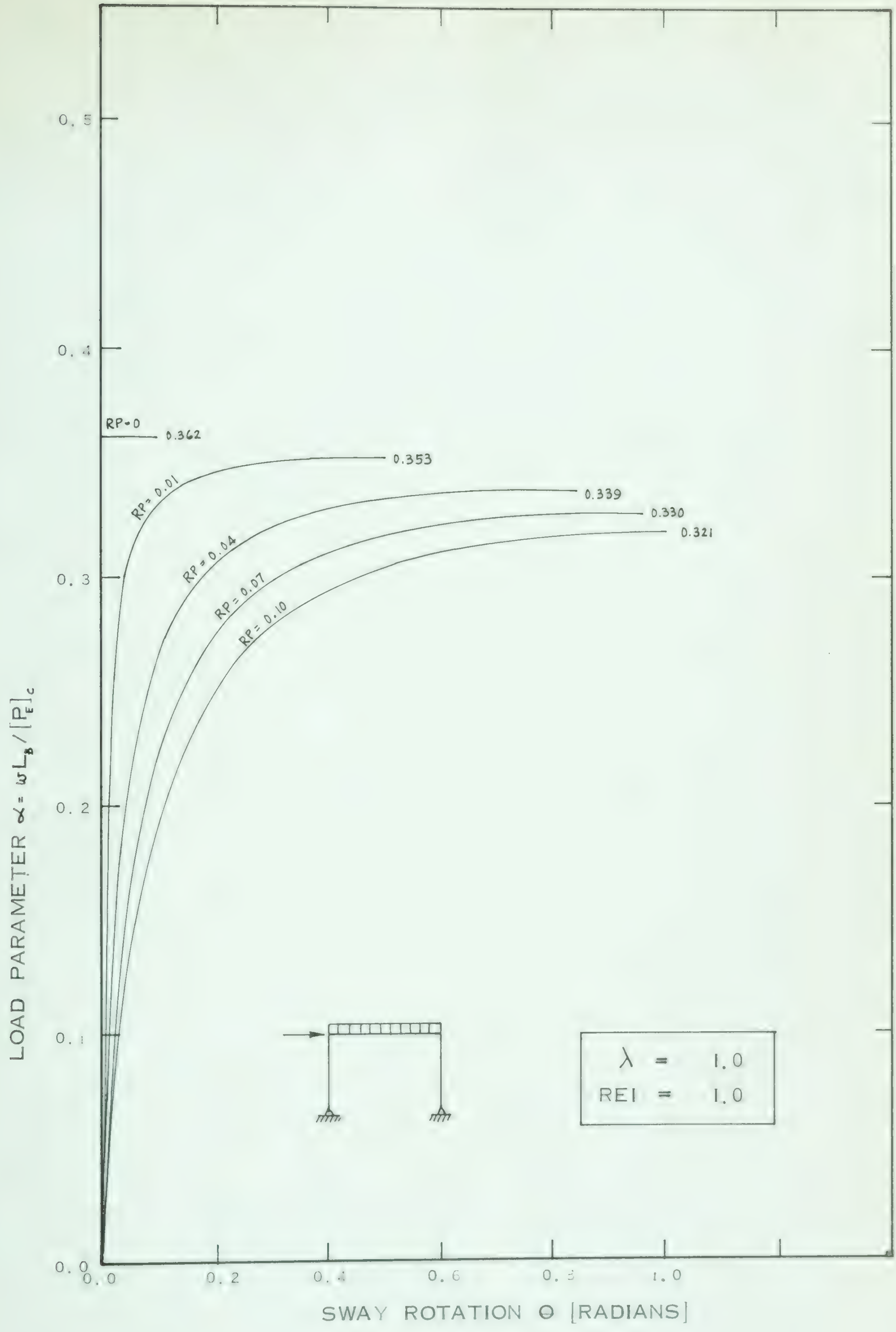


FIGURE 4 – 3 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

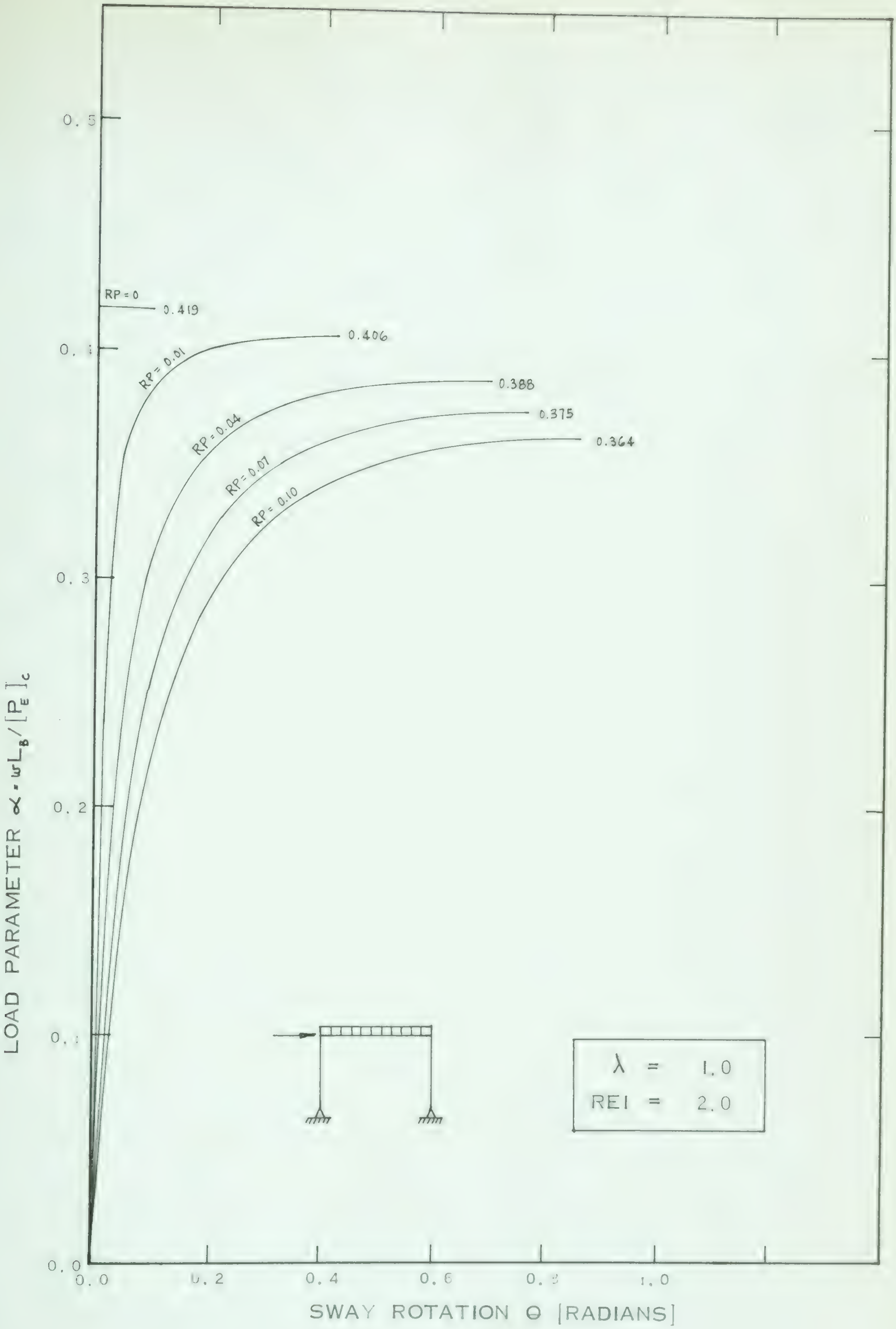


FIGURE 4 – 4 LOAD – DEFLECTION – SMALL DEFLECTION THEORY

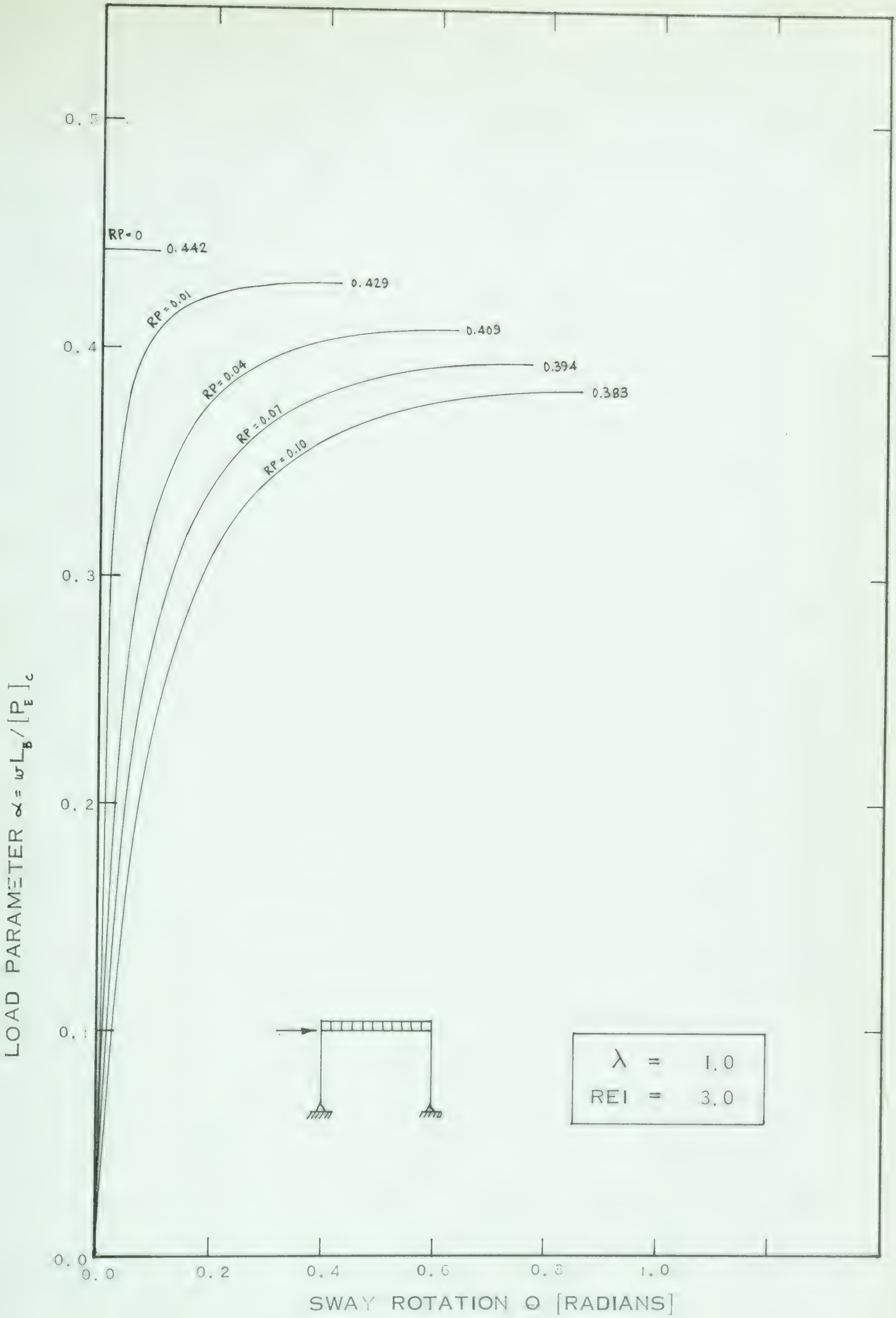


FIGURE 4 – 5 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

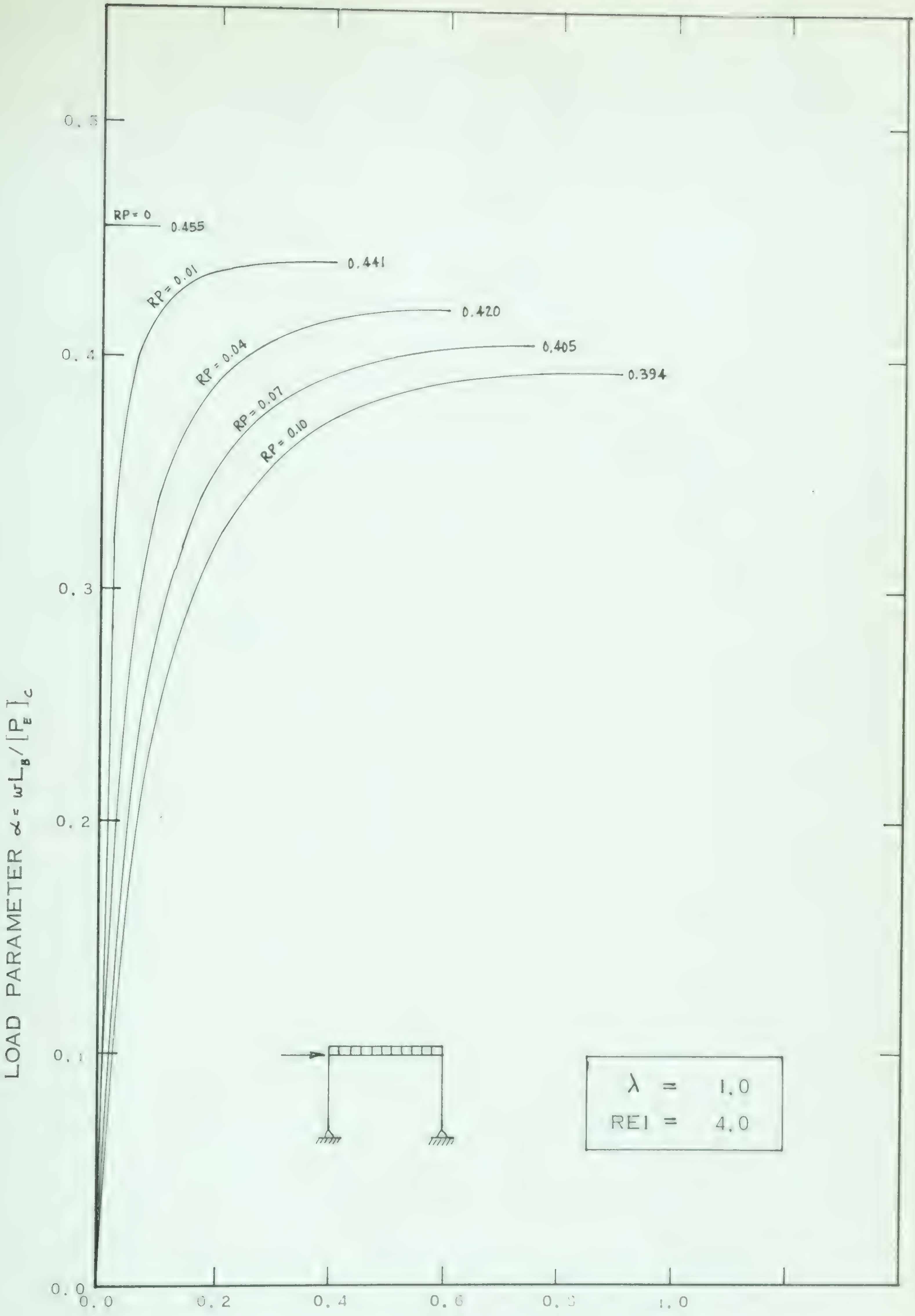


FIGURE 4 – 6 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

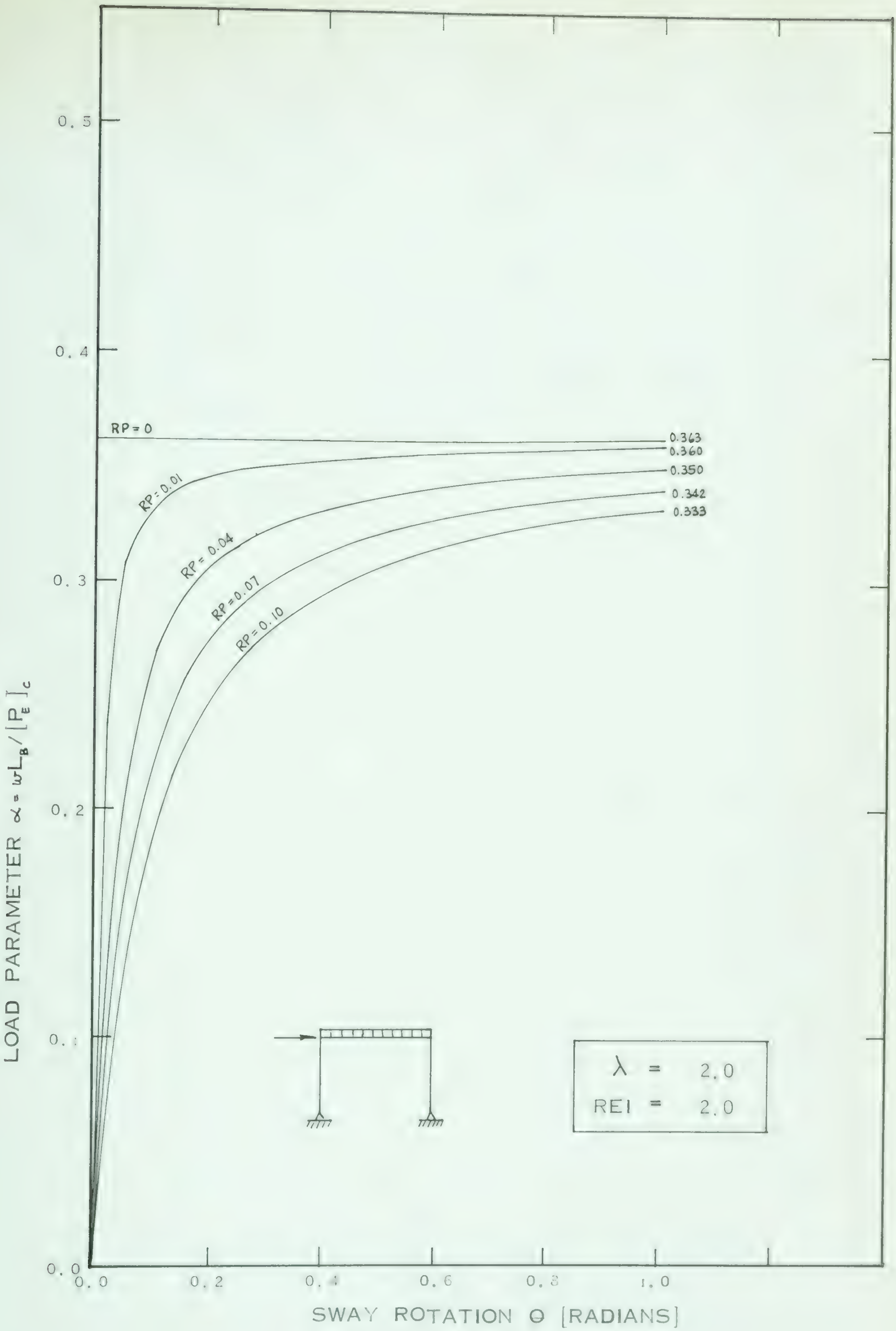


FIGURE 4-7 LOAD vs. DEFLECTION - SMALL DEFLECTION THEORY

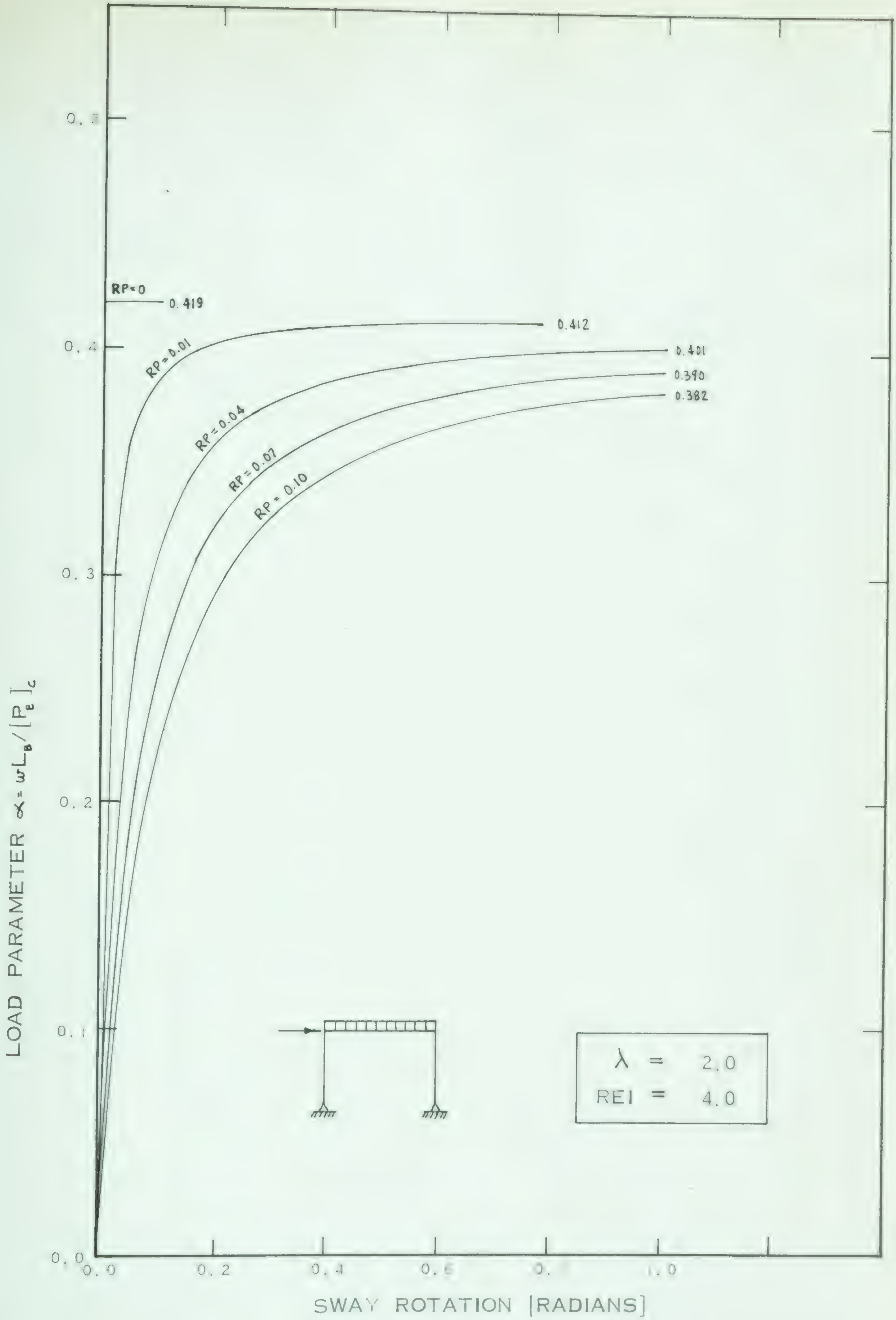


FIGURE 4 – 8 LOAD vs. DEFLECTION – SMALL DEFLECTION THEOR

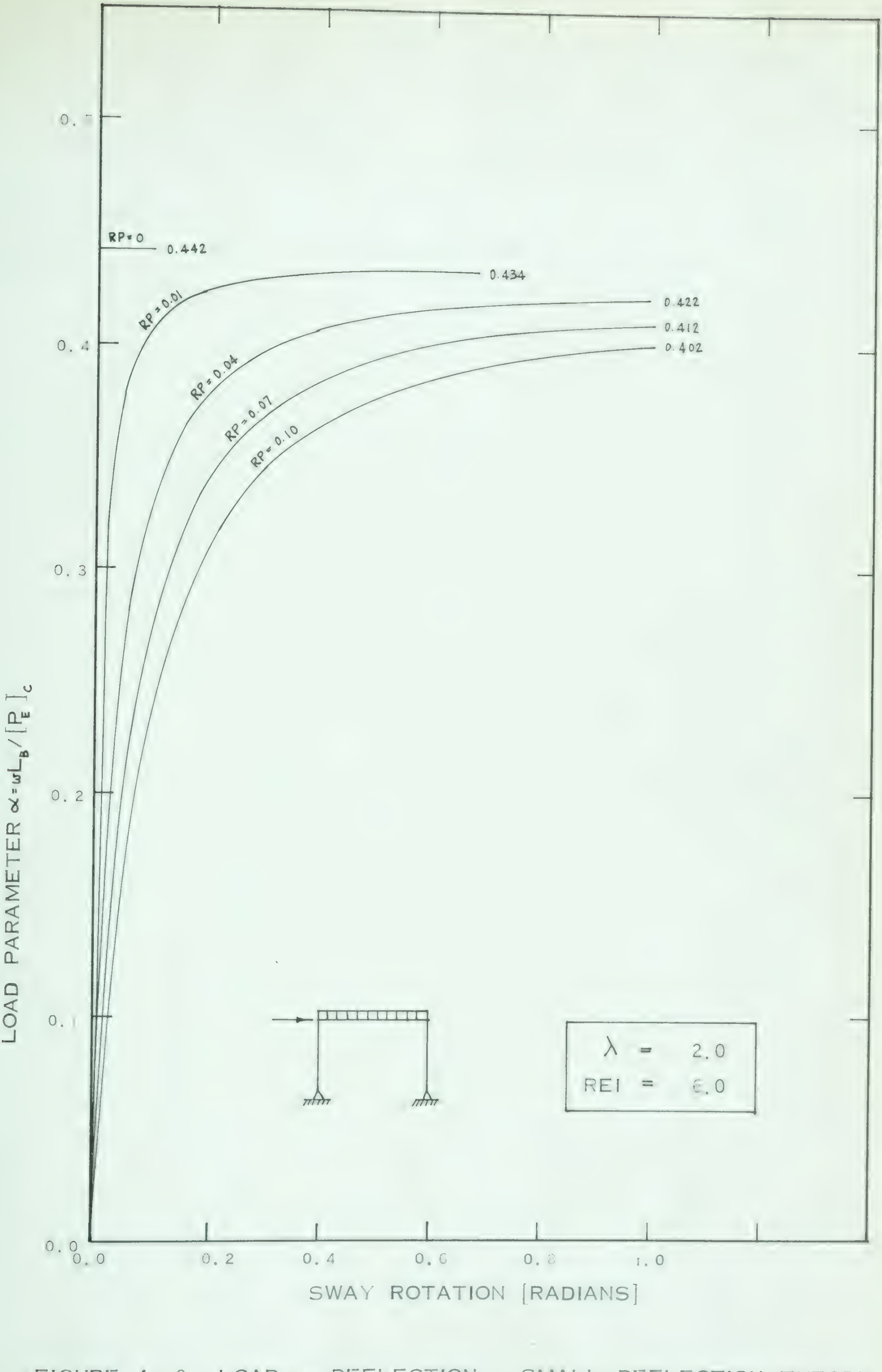


FIGURE 4 – 9 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

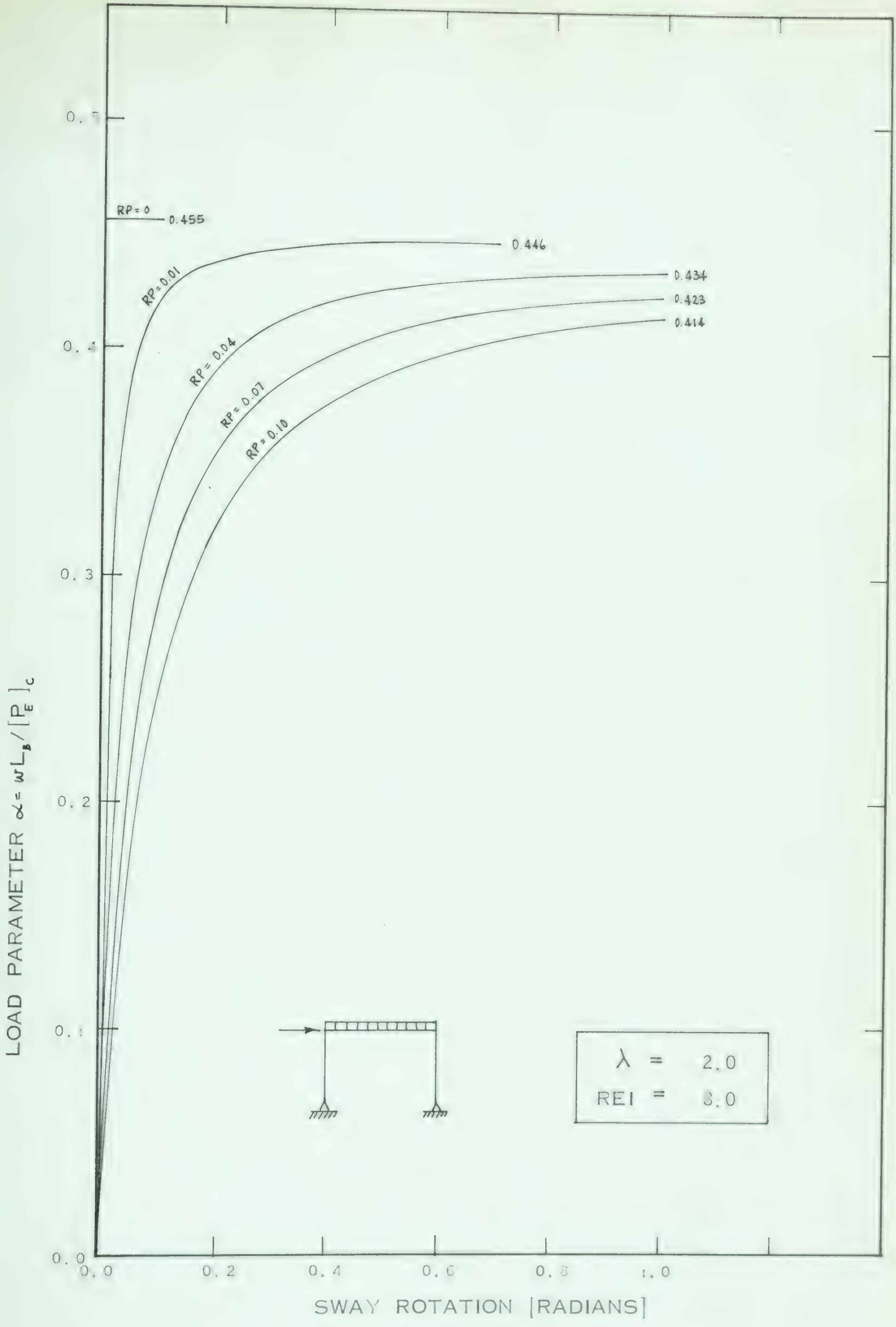


FIGURE 4 – 10 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

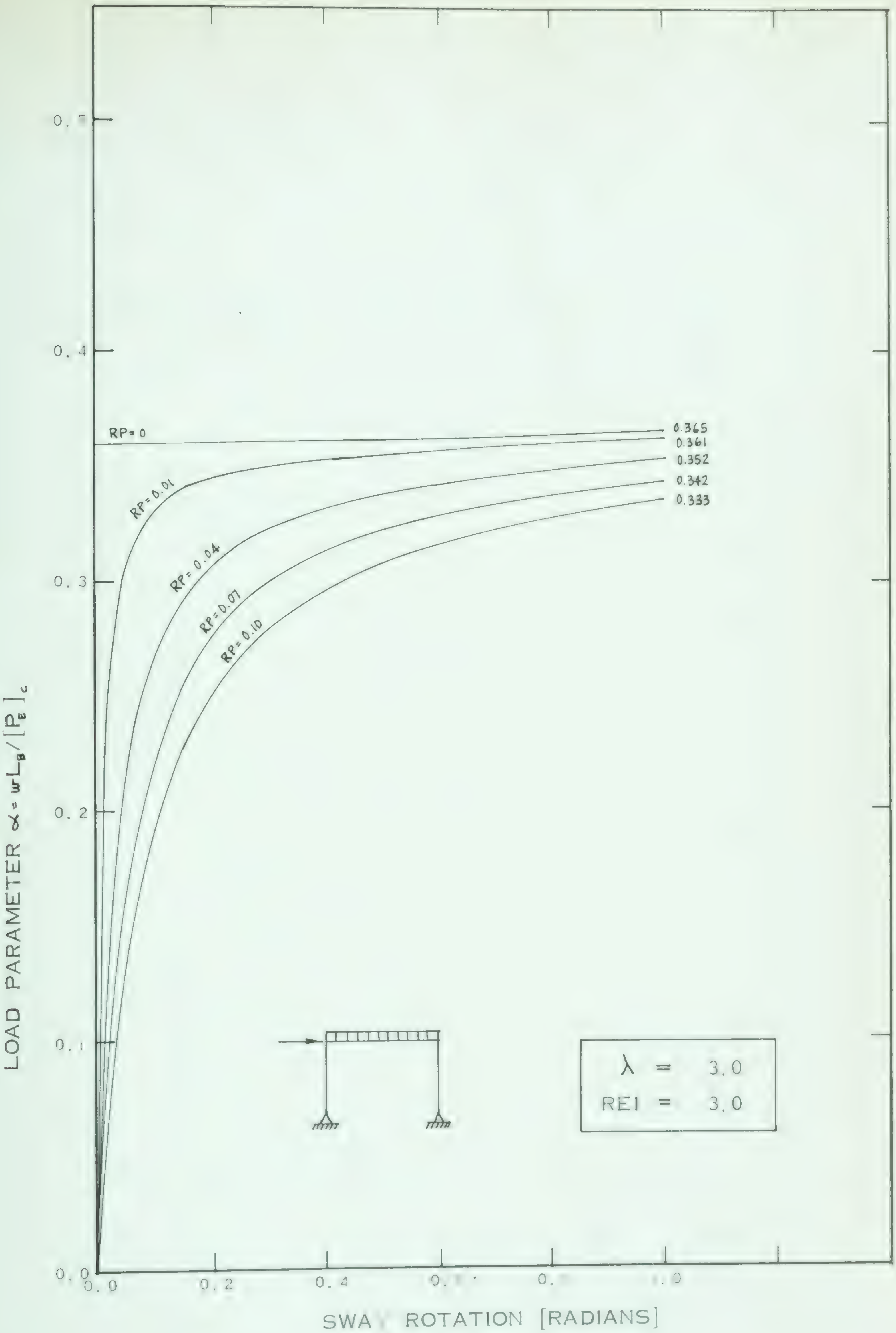


FIGURE 4 – II LOAD vs. DEFLECTION SMALL DEFLECTION THEOR

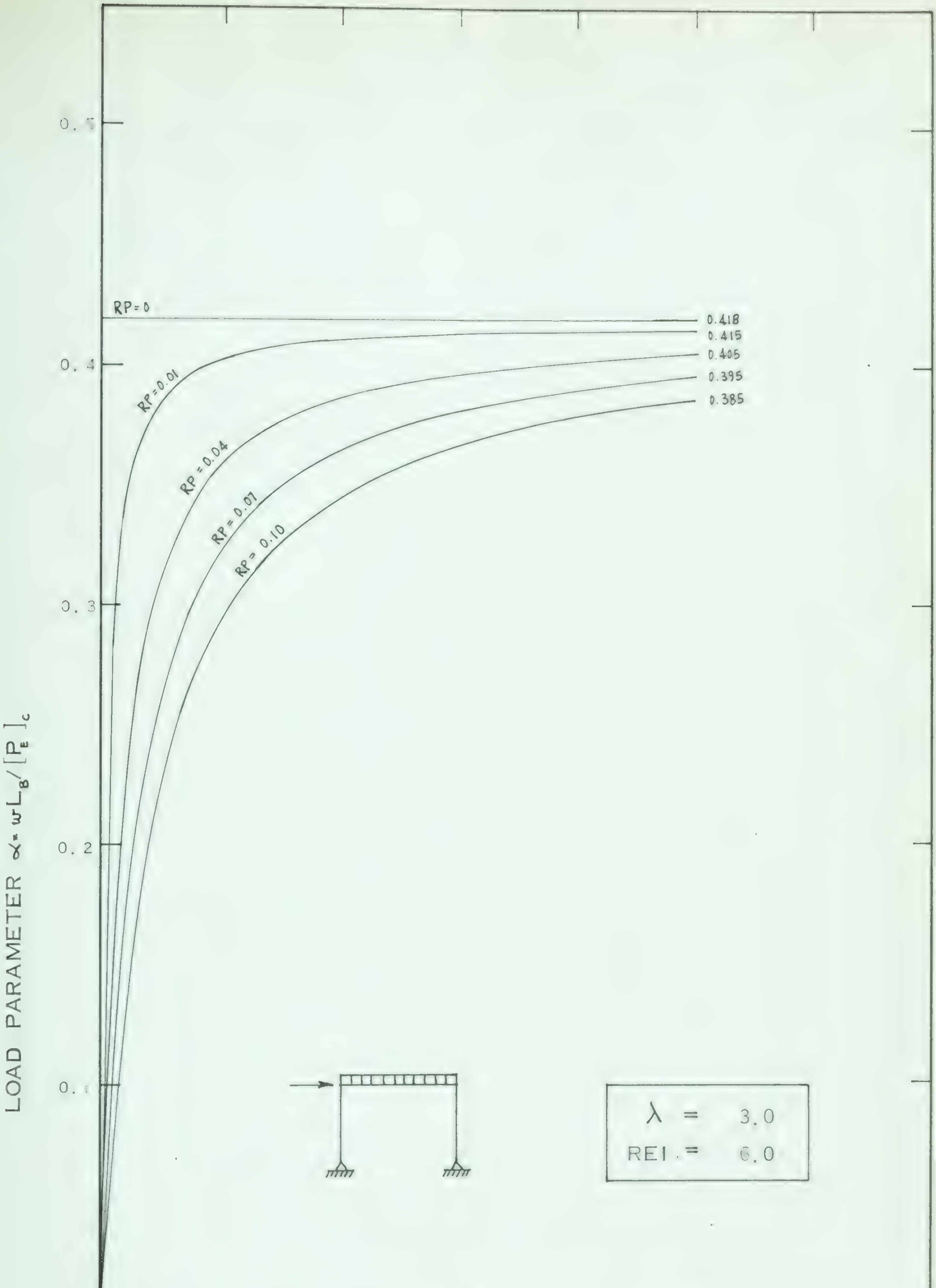


FIGURE 4 – 12 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

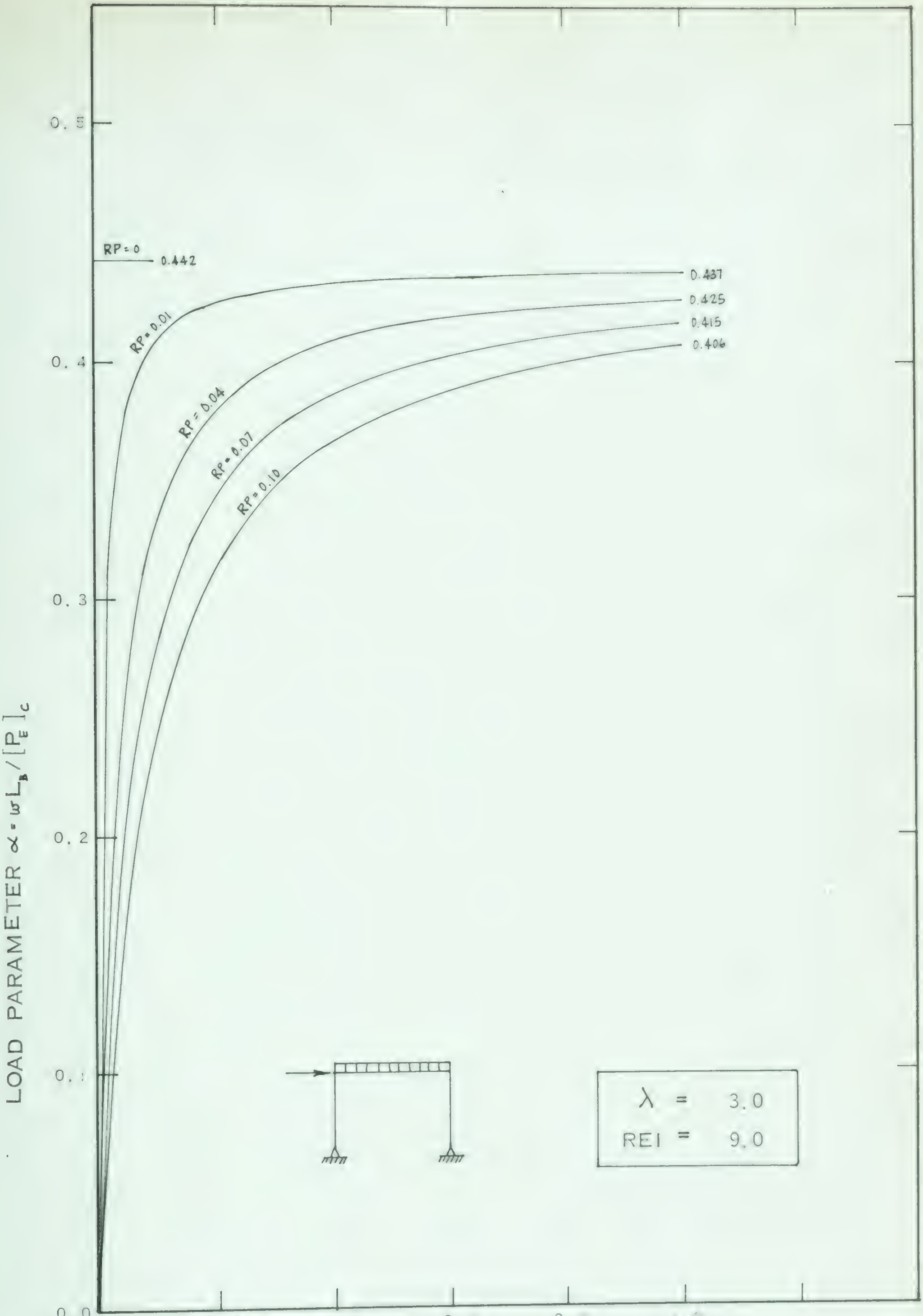


FIGURE 4 – 13 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

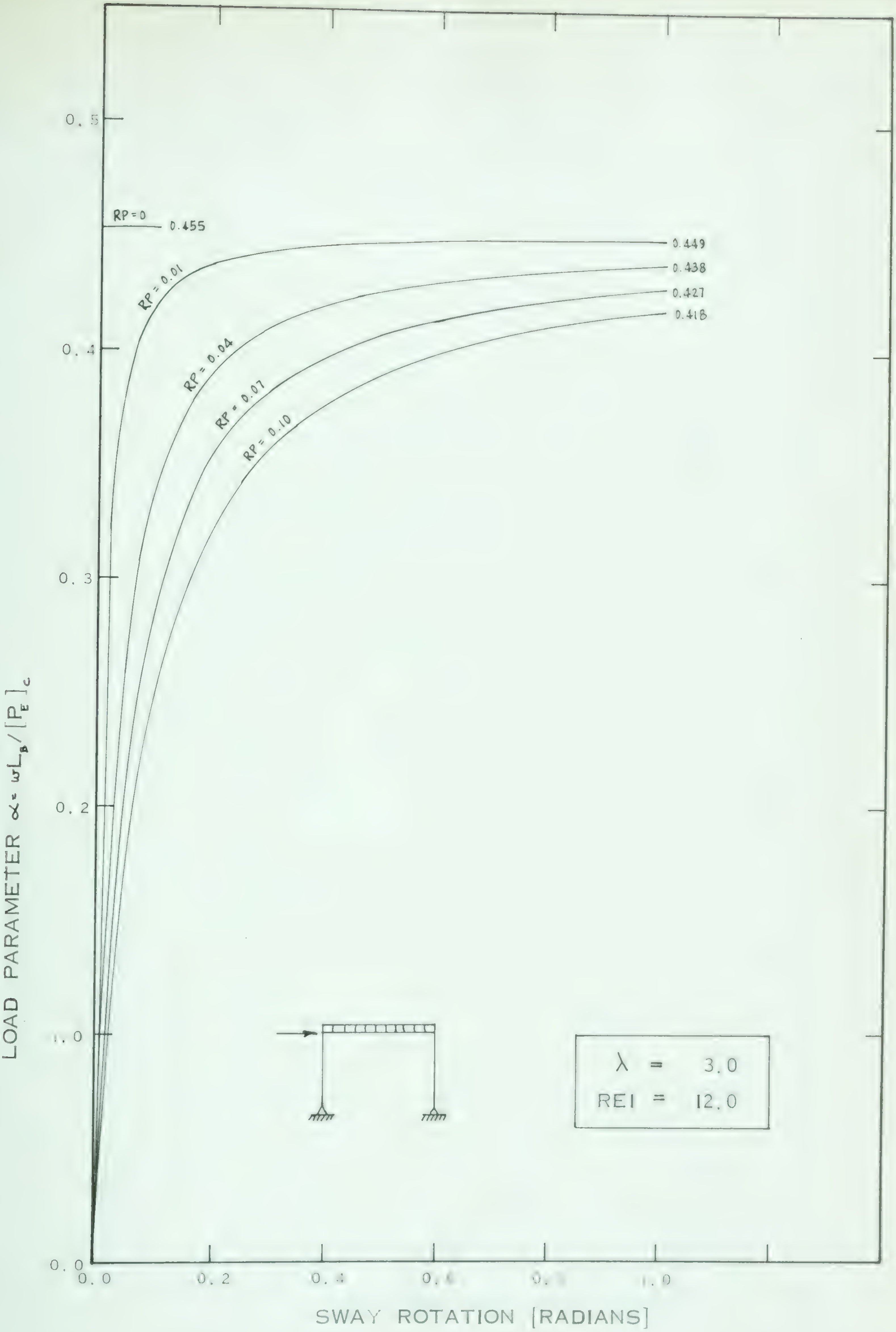


FIGURE 4 – 14 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

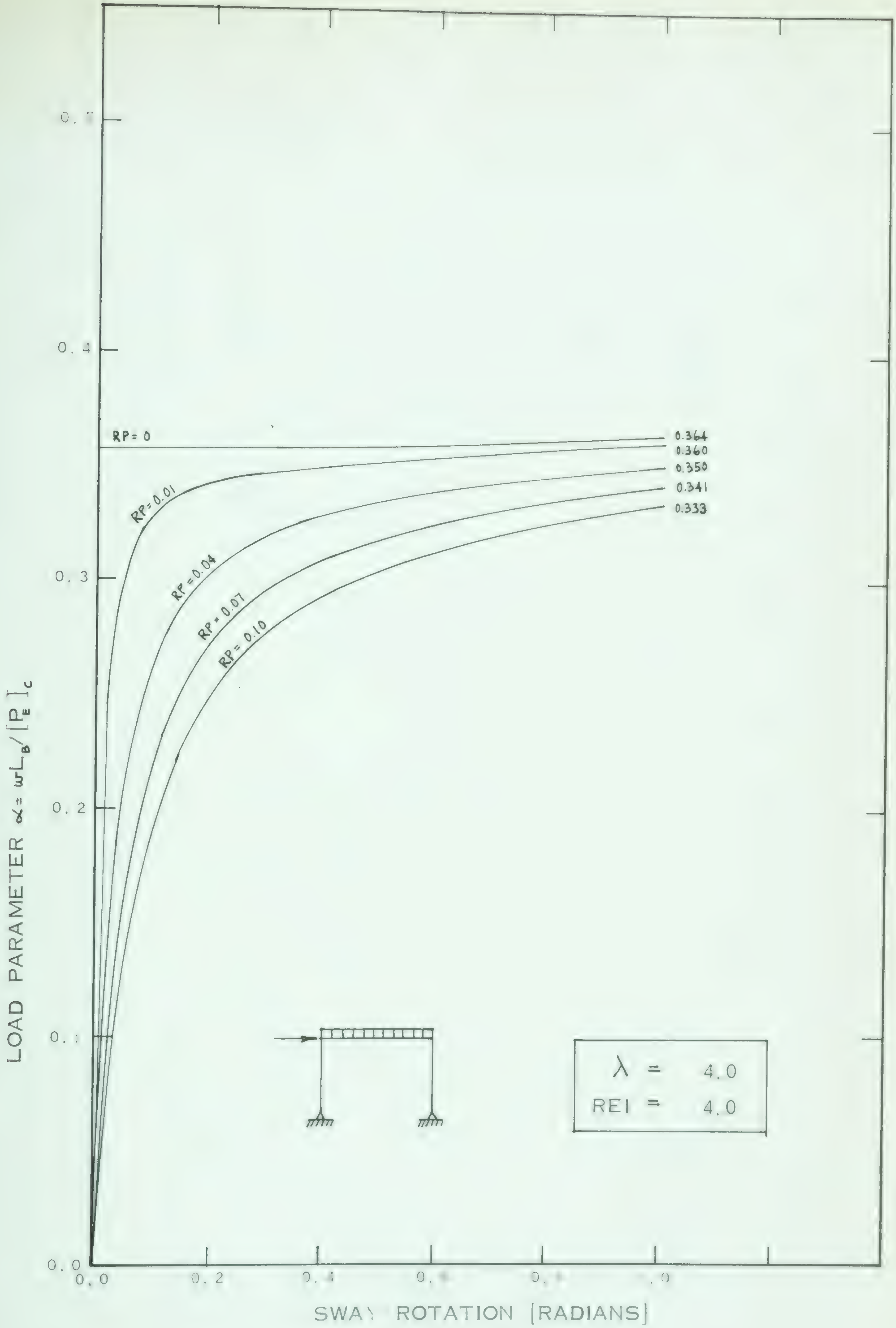


FIGURE 4 – 15 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

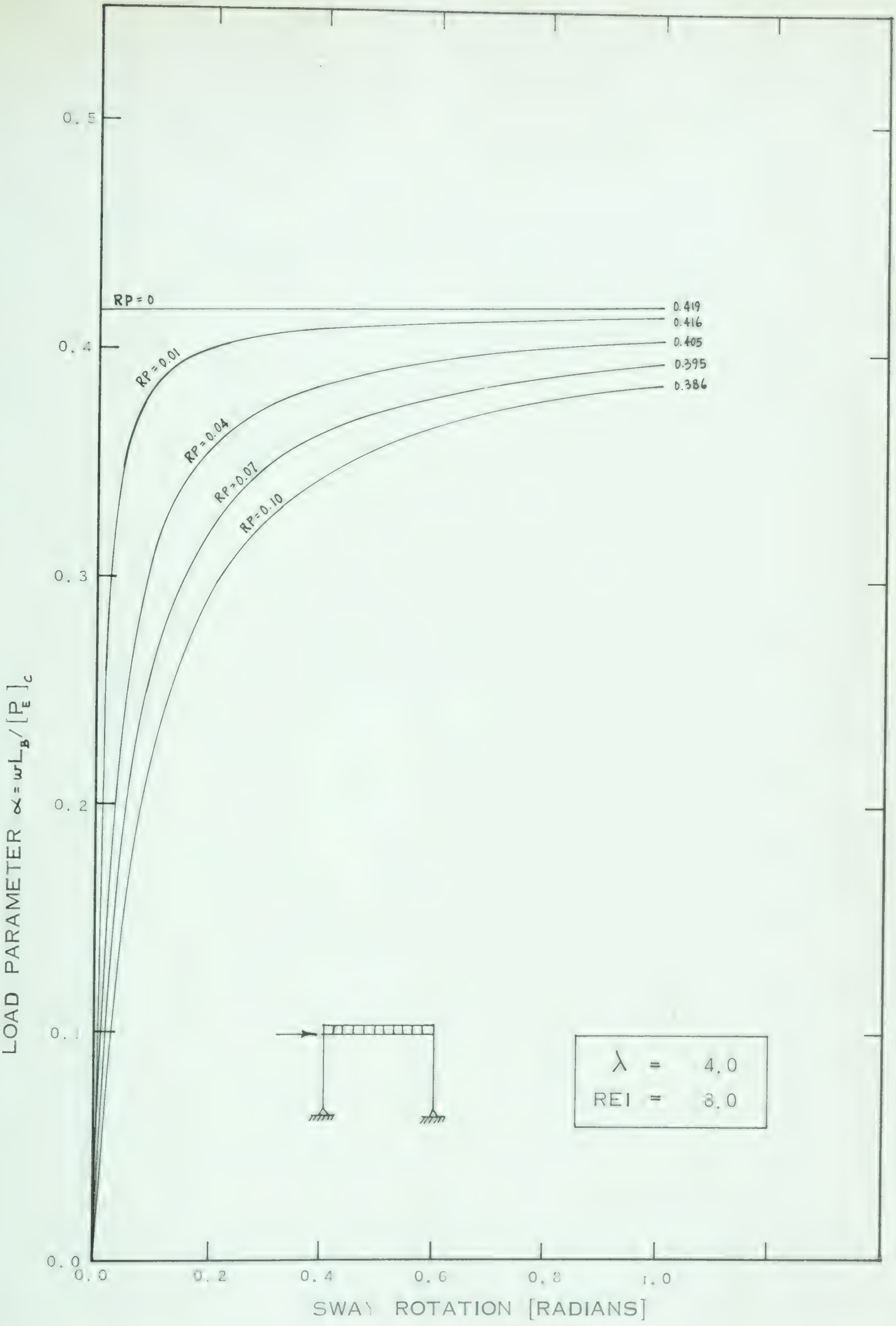


FIGURE 4 – 16 LOAD . DEFLECTION – SMALL DEFLECTION THEOR

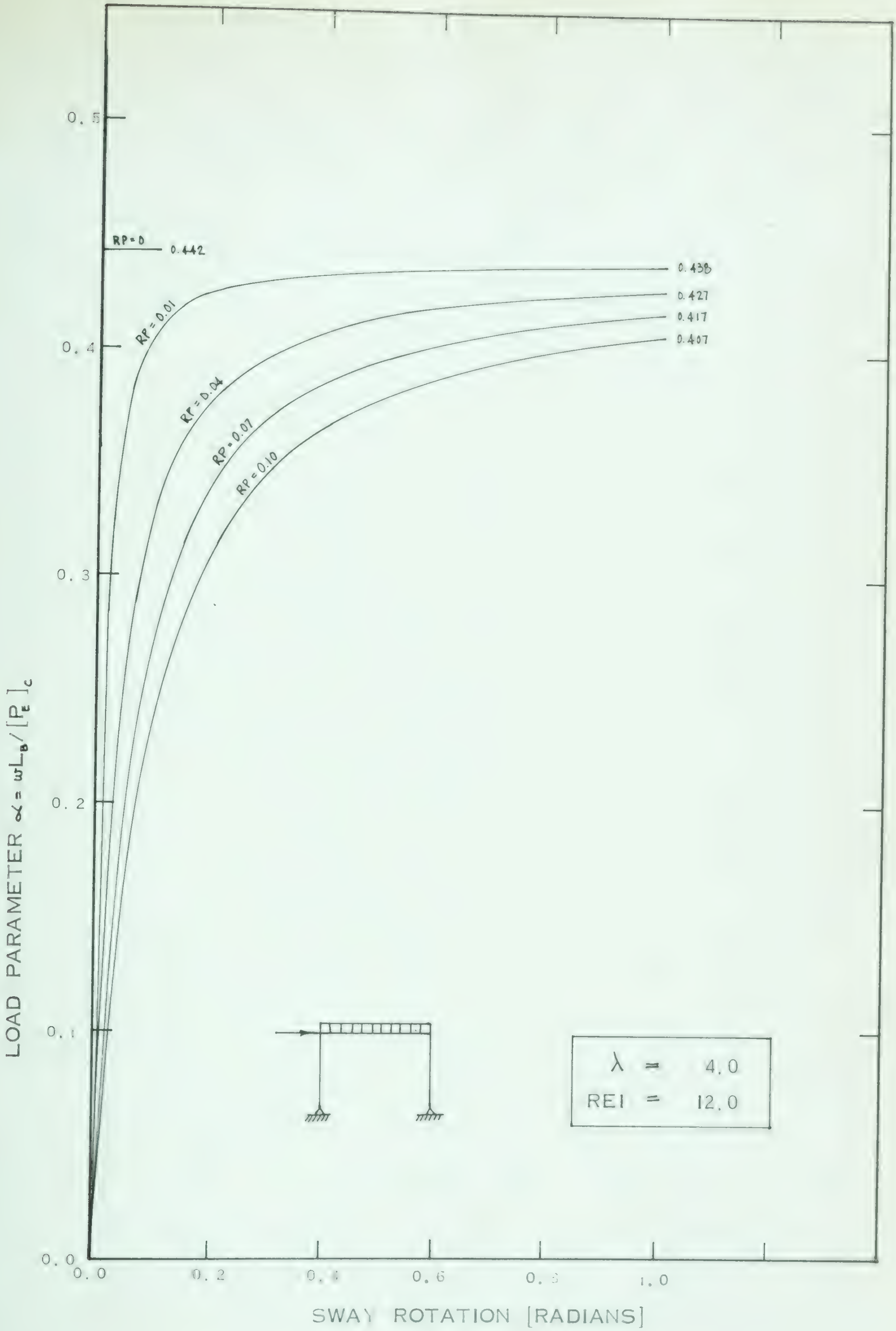


FIGURE 4 – 17 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

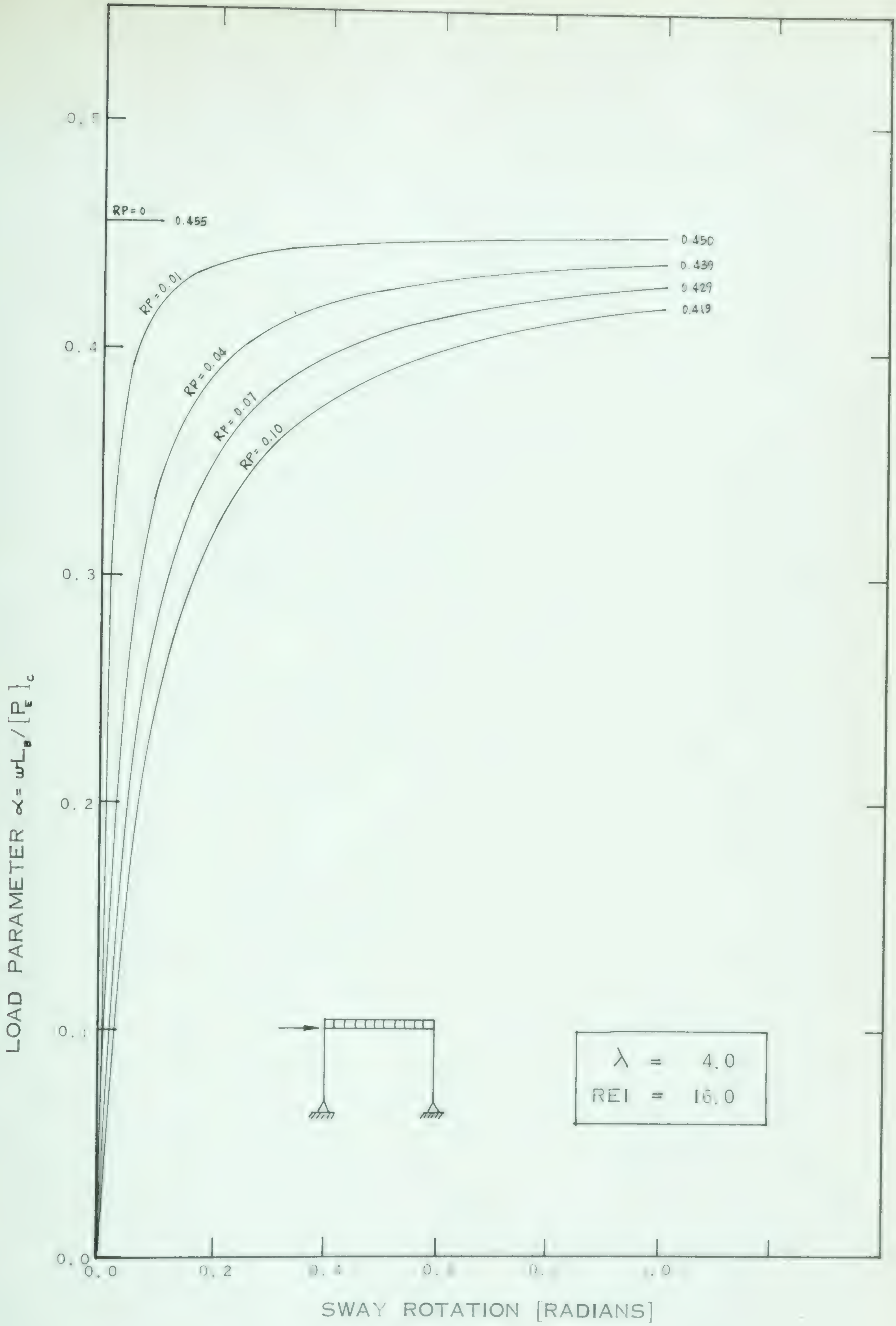


FIGURE 4 – 18 LOAD – DEFLECTION – SMALL DEFLECTION THEORY

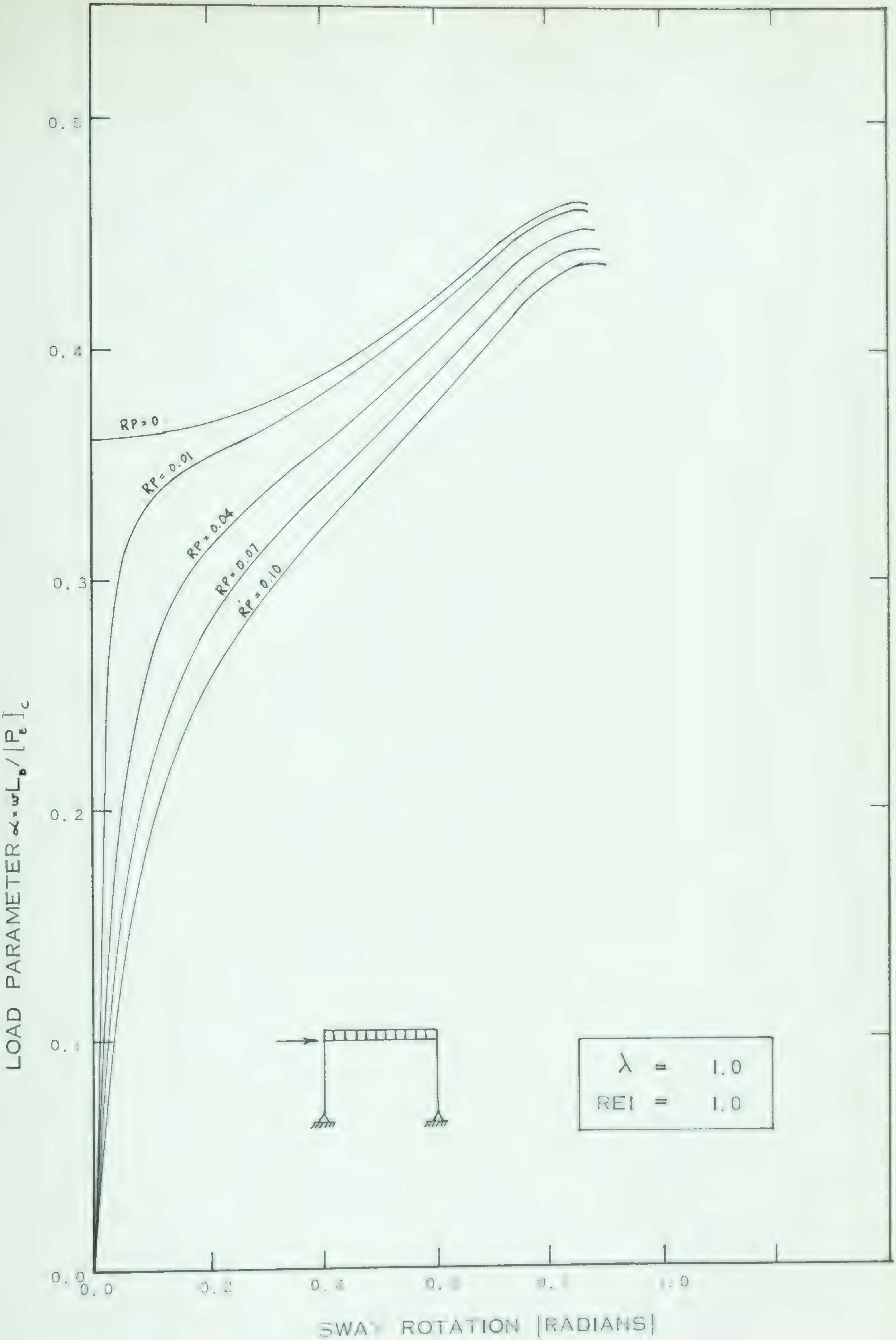


FIGURE 4 – 19 LOAD vs. DEFLECTION LARGE DEFLECTION THEORY

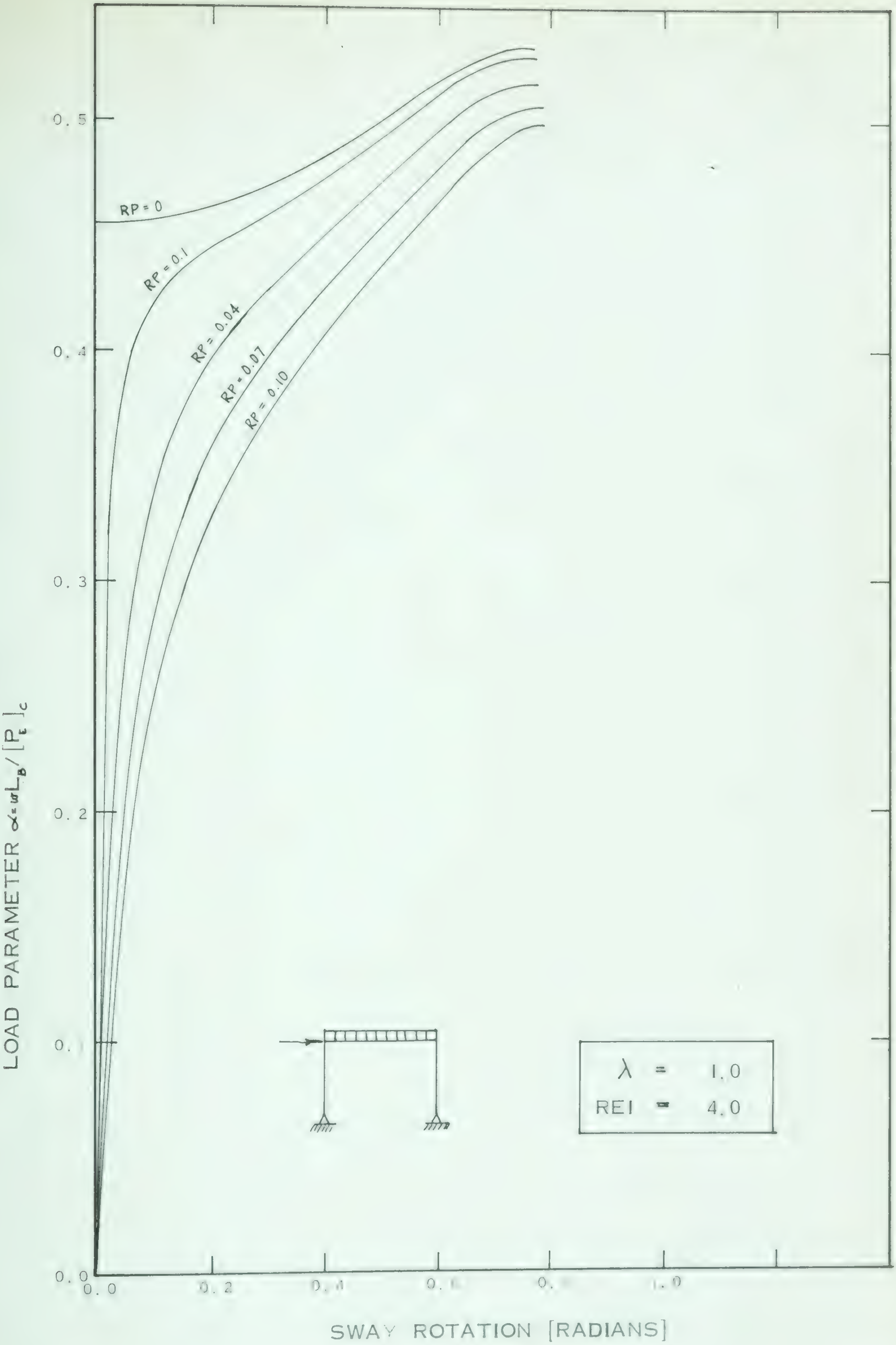


FIGURE 4 – 20 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

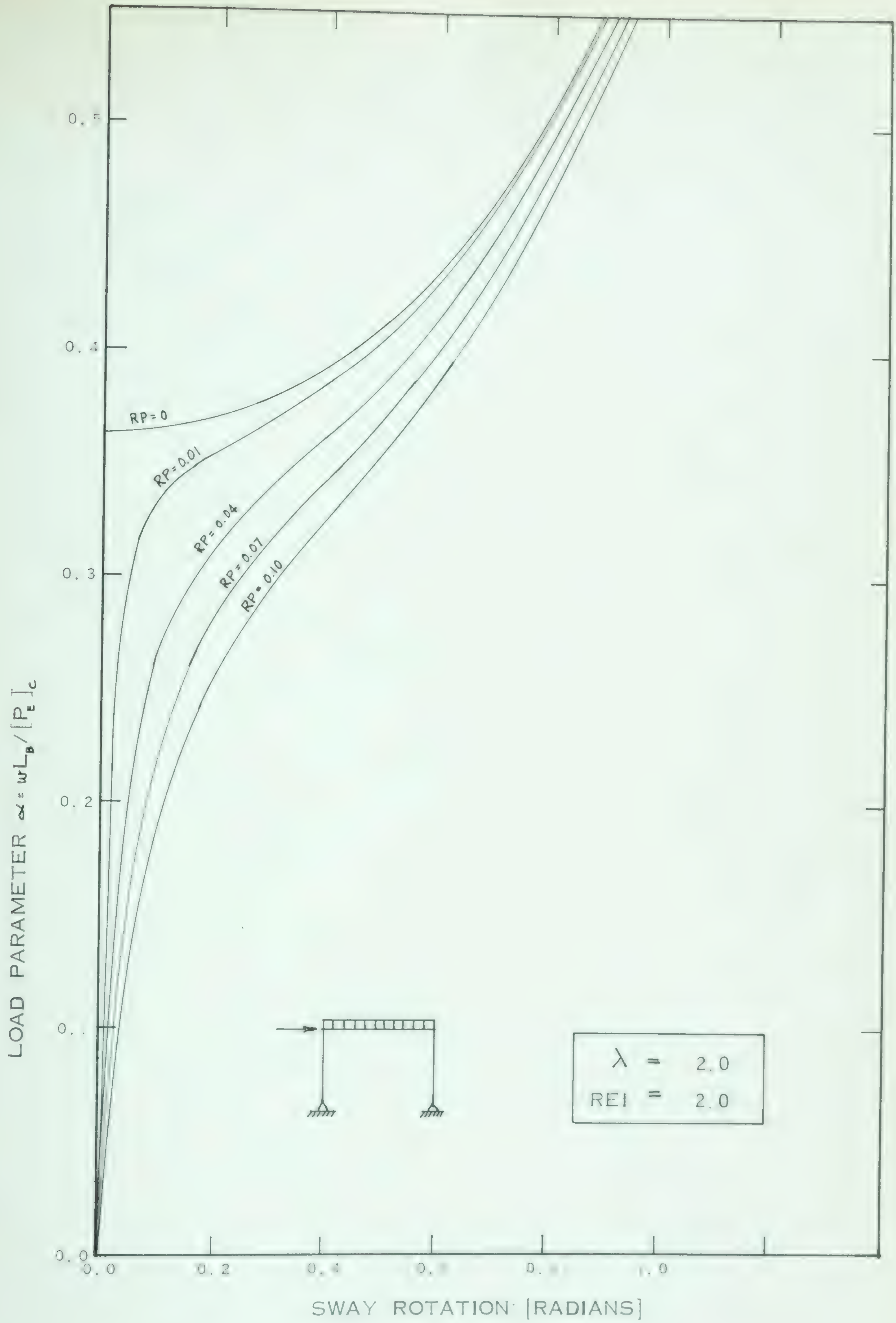


FIGURE 4 – 21 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

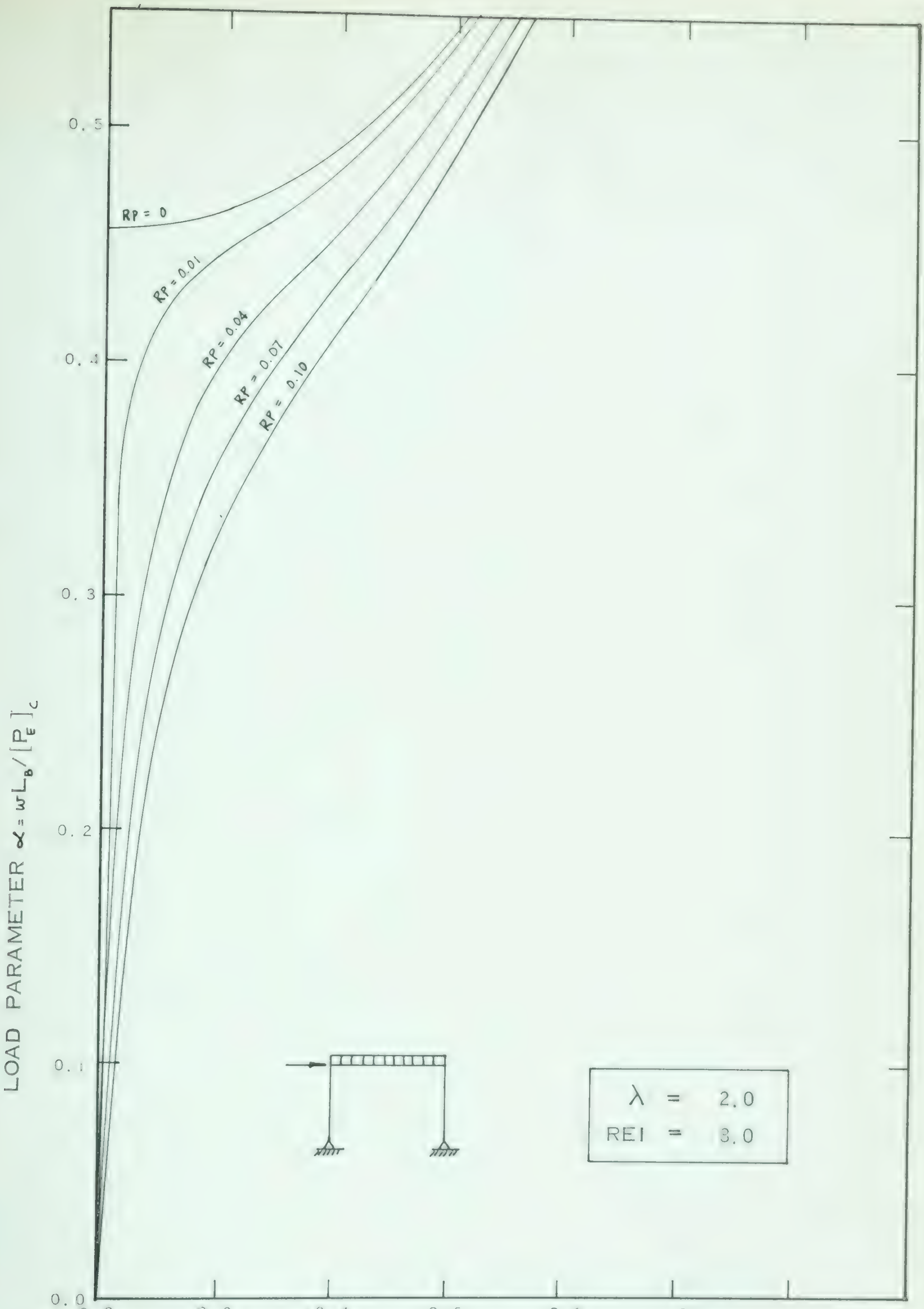


FIGURE 4 – 22 LOAD ... DEFLECTION – LARGE DEFLECTION THEOR

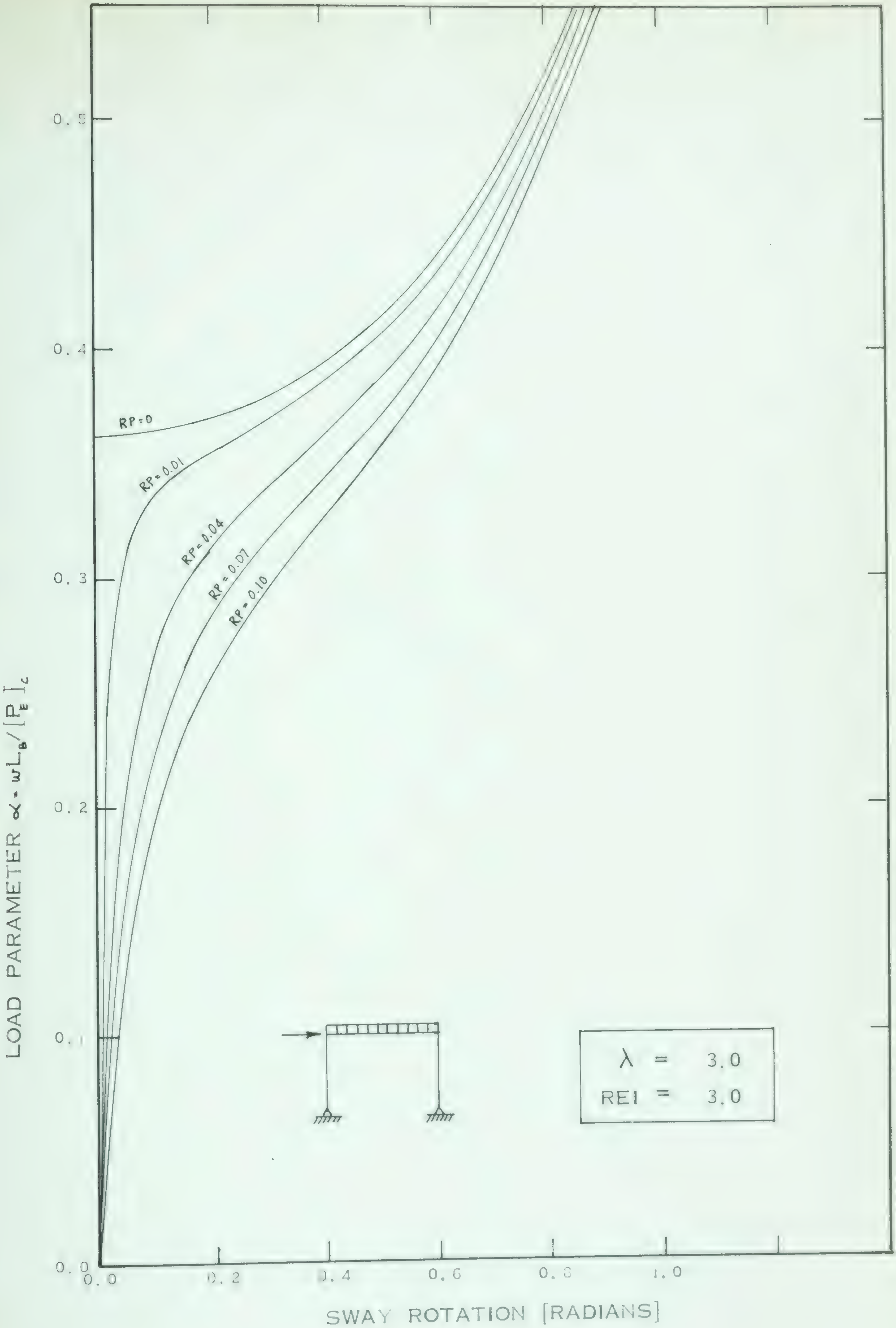


FIGURE 4-23 LOAD vs. DEFLECTION - LARGE DEFLECTION THEORY

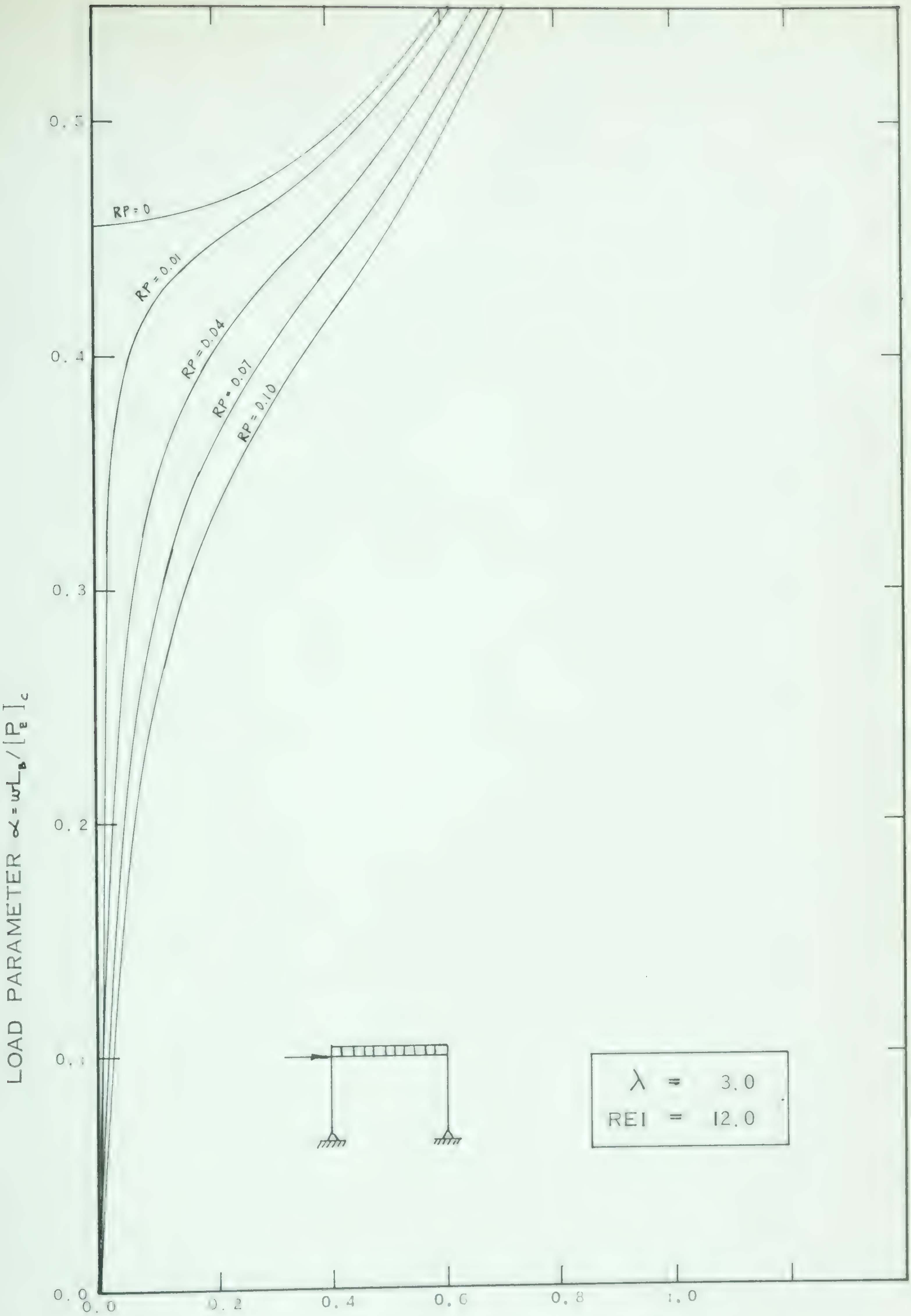


FIGURE 4 – 24 LOAD vs. DEFLECTION – LARGE DEFLECTION THEOR

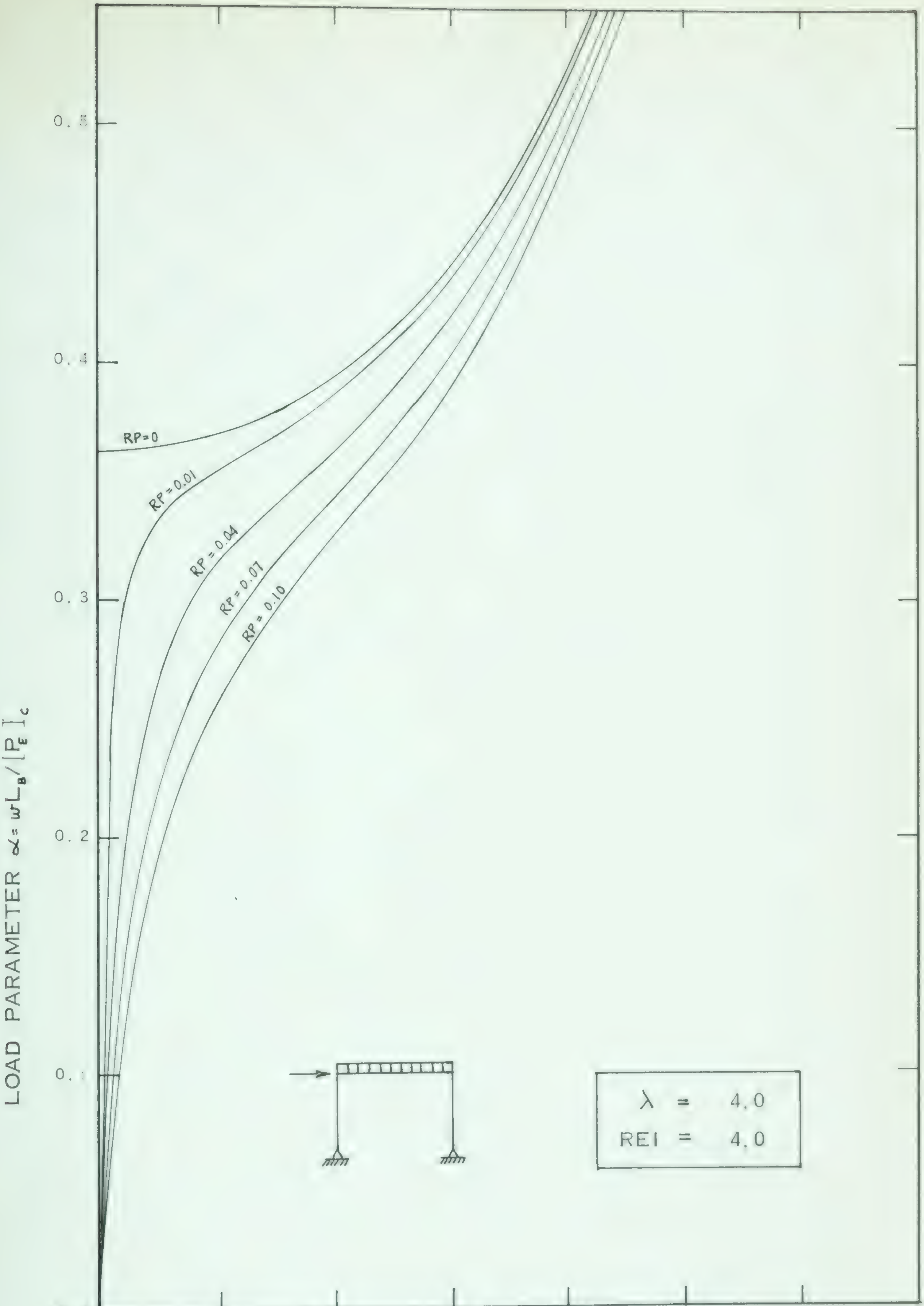


FIGURE 4 – 25 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

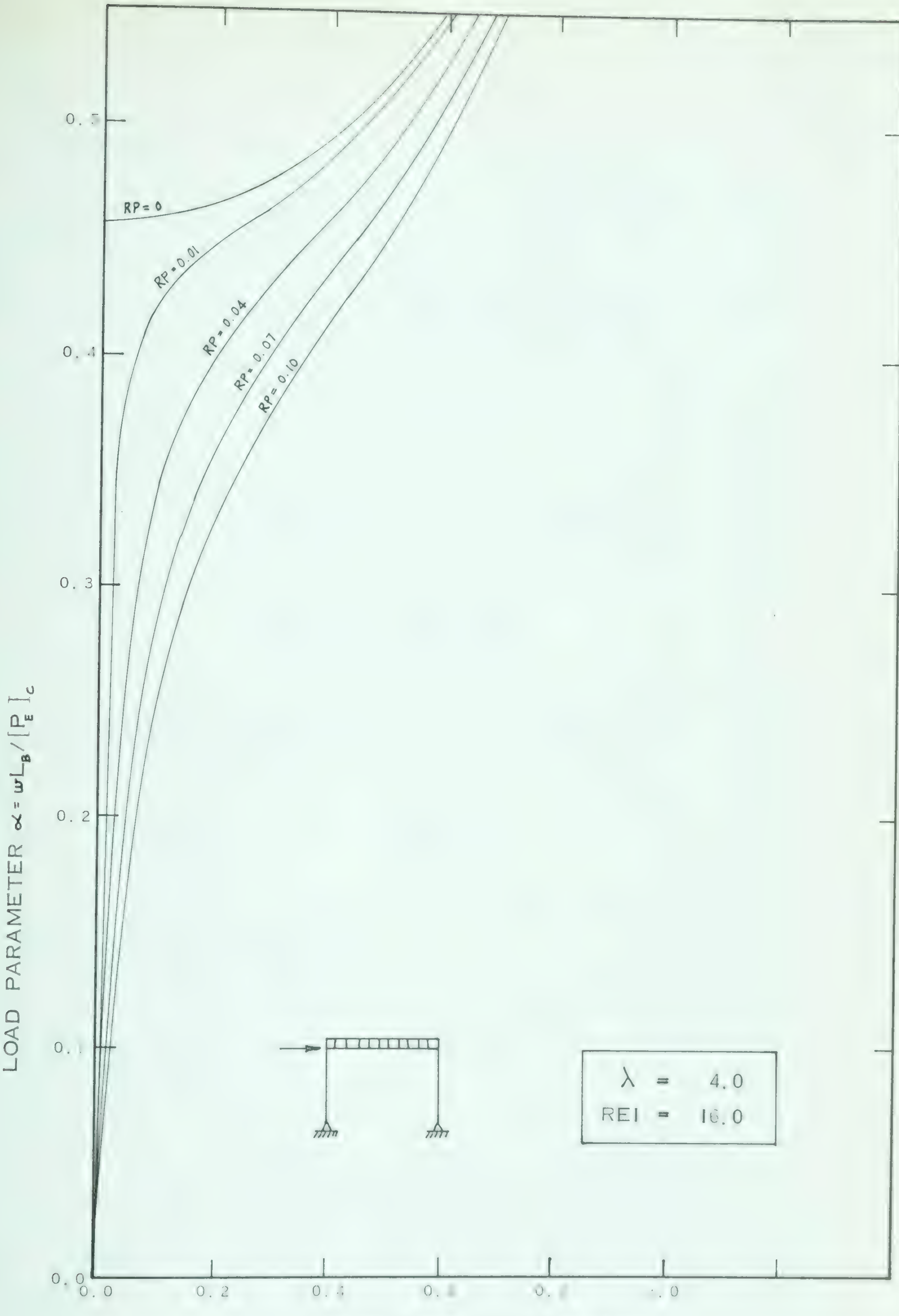


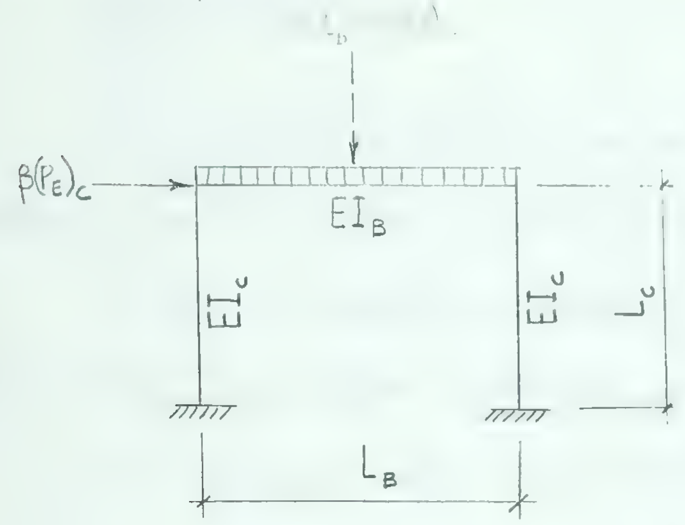
FIGURE 4 – 26 LOAD vs. DEFLECTION – LARGE DEFLECTION THEOR

λ	REI →	1.0	2.0	3.0	4.0	6.0	8.0	9.0	12.0	16.0
1.0	RP = 0	0.362*	0.419	0.442	0.455					
	RP = 0.01	0.353	0.406	0.429	0.441					
	RP = 0.04	0.339	0.388	0.409	0.420					
	RP = 0.07	0.330	0.375	0.394	0.405					
	RP = 0.10	0.321	0.364	0.383	0.394					
2.0	RP = 0		0.363		0.419	0.442	0.455			
	RP = 0.01		0.360		0.412	0.434	0.446			
	RP = 0.04		0.350		0.401	0.422	0.434			
	RP = 0.07		0.342		0.390	0.412	0.423			
	RP = 0.10		0.333		0.382	0.402	0.414			
3.0	RP = 0			0.365		0.418		0.442	0.455	
	RP = 0.01			0.361		0.415		0.437	0.449	
	RP = 0.04			0.352		0.405		0.425	0.438	
	RP = 0.07			0.342		0.395		0.415	0.427	
	RP = 0.10			0.333		0.385		0.406	0.418	
4.0	RP = 0				0.364		0.419		0.442	0.455
	RP = 0.01				0.360		0.416		0.438	0.450
	RP = 0.04				0.350		0.405		0.427	0.439
	RP = 0.07				0.341		0.395		0.417	0.429
	RP = 0.10				0.333		0.386		0.407	0.419

*Values in Terms of = $wL_B/(P_E)_C$

Table of Elastic Critical Loads for Frame with Hinged Bases

4-3 FRAME WITH FIXED BASES



$RP = \beta/\alpha$

$\lambda = L_B/L_C$

$REI = EI_B/EI_C$

FIGURE 4-27

The relationships between the load parameter α and sway rotation θ are shown in FIGURES 4-29 to 4-52 for five values of horizontal load considered. The results of the large-deflection theory are given following those of the small deflection theory.

For the case of no horizontal load, the maximum vertical load the frame can support if it remains elastic is $2.0(P_E)_c$, which occurs with an infinitely stiff beam member. This can be deduced by considering one of the column members in its buckled configuration under this condition, as in FIGURE 4-28.

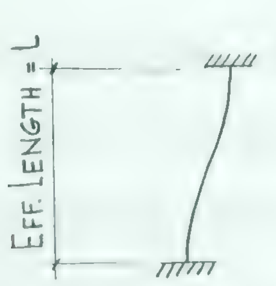


FIGURE 4-28

The effective length of this column member is 1.0 L, making the

elastic critical load equal to

$$\frac{\pi^2 EI}{L^2} = 1.0 \; P_E$$

The frame has two such column supports, and hence its elastic critical load becomes $2 \times 1.0 \; (P_E)_c = 2.0 \; (P_E)_c$.

It can be seen by considering the results of the small deflection theory that the maximum vertical load sustained by this frame with no horizontal load present does in fact approach $2.0 \; (P_E)_c$ as the beam stiffness increases. Conversely, when the beam stiffness is relatively small, the critical load drops considerably since the effective lengths of the columns are free to increase above L.

Difficulty was encountered with some of the cases in this series. For some of the cases where λ was relatively large and REI relatively small ($\lambda = 3.0$, REI = 3.0 and 6.0, and $\lambda = 4.0$, REI = 4.0, 8.0, 12.0), the computer results were very erratic as the critical load was approached. In some cases, it was possible only to complete the lower portions of the load deflection plots. For the remaining graphs, the results were extrapolated from the partial plots and the complete results of other graphs. Any extrapolated curves are shown as broken lines instead of solid ones. The reasons for this difficulty are not definitely known, but it is believed the mathematical solution becomes unstable as the critical load is approached for these particular combinations of variables. It is assuring that these difficulties have occurred in the less practical portion of the range of variables considered, since it is probable that larger beam stiffnesses would be used when λ is large.

As in the case of the pinned base frame, the existence of even a small horizontal load causes a significant increase in sway rotations, with

larger sway rotations being evident for larger horizontal loads. Again, the effect of horizontal loads on critical vertical loads was difficult to assess quantitatively, since a definite critical load was not always reached within a range of sway rotations less than 1.0. On the basis of the results of the small deflection theory, it may be stated in general that the existence of a horizontal load does not decrease the critical load substantially, but has more effect on sway rotations.

The results of a "large-deflection theory" for only the extremes of the variables presented for the small-deflection theory are given in Figures 4-45 to 4-52. As for the pinned-base frame, the "large-deflection theory" produced higher loads than the small-deflection theory in the range of large sway rotations, whereas the results are identical for very small sway rotations.

As discussed for the hinged-base frame, the use of the equation $M = EId^2y/dx^2$ introduces errors to both the small and large deflection theories used in this thesis. Allowing an error of 5% between this equation and the corresponding exact equation, and investigating the computer results for this frame, it was found that an error of this magnitude will occur in some cases for sway rotations as small as $\theta = 0.05$ radians. Therefore, the results of both the large and small deflection theories must be considered inaccurate for larger sway rotations.

Considering the results of the small-deflection theory as a reasonably conservative estimate of the elastic critical loads of this frame, the same general observations as for the pinned base frame can be made:

(1) The elastic critical load decreases with increasing horizontal load, all other variables held constant. The percentage reduction from the case of no horizontal load is about the same for all combinations of λ and REI investigated.

(2) The elastic critical load decreases with increasing values of λ, the variables RP and REI being held constant.

(3) The elastic critical load increases with increasing values of REI, the variables RP and λ being held constant, i.e., the critical load increases with increasing beam stiffness.

A summary of elastic critical loads obtained is given in tabular form following the graphs.

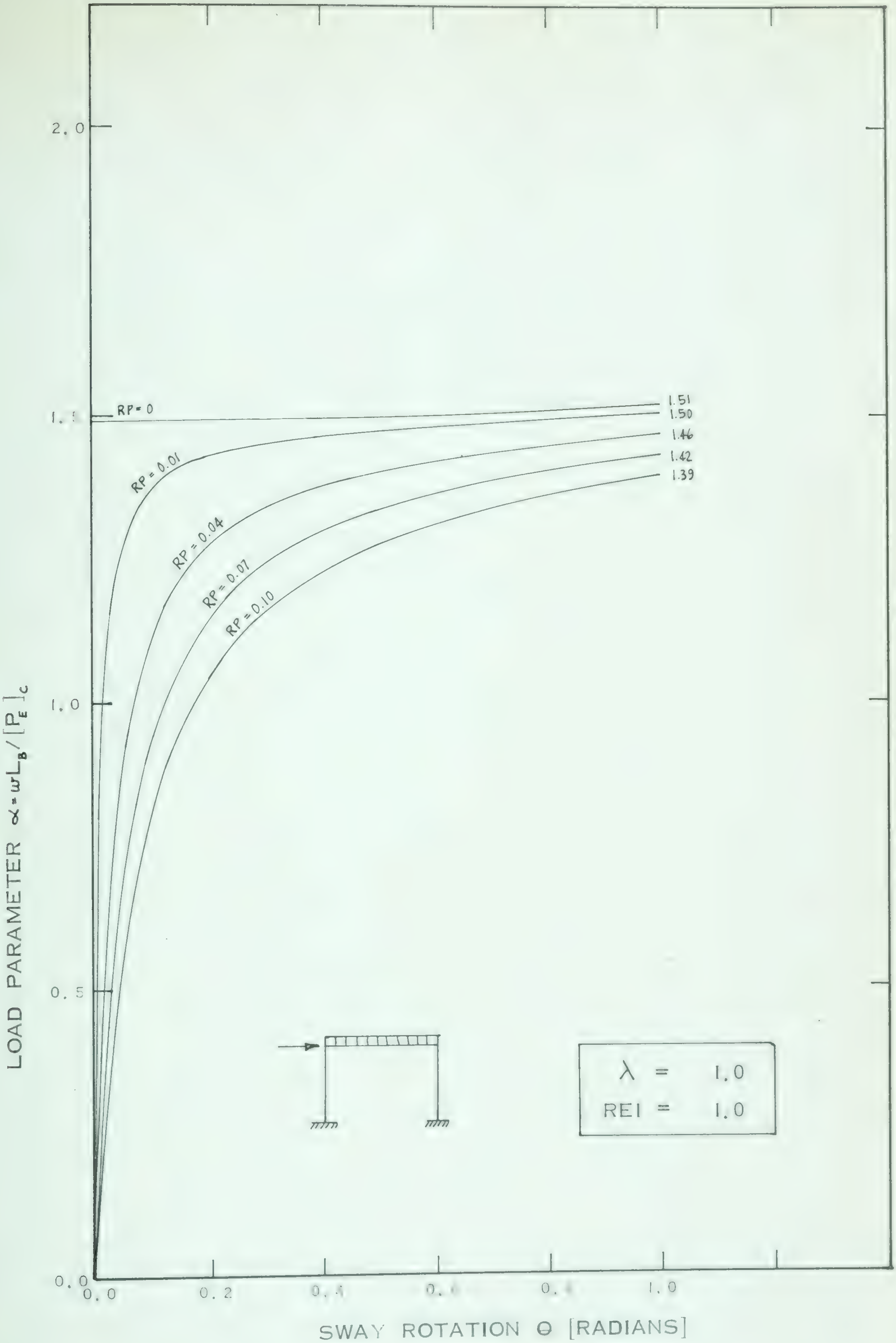


FIGURE 4-29 LOAD vs. DEFLECTION - SMALL DEFLECTION THEORY

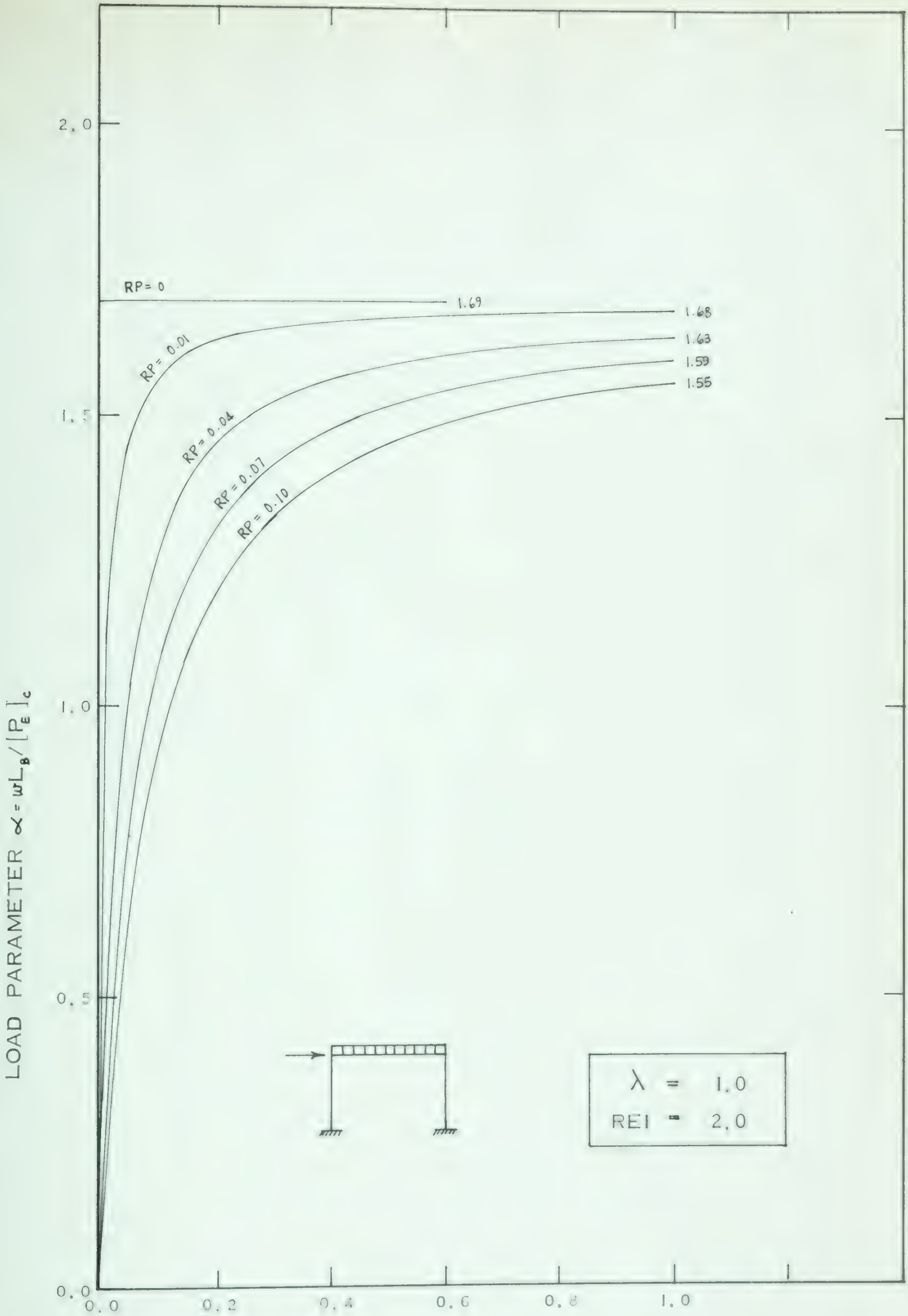


FIGURE 4 – 30 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

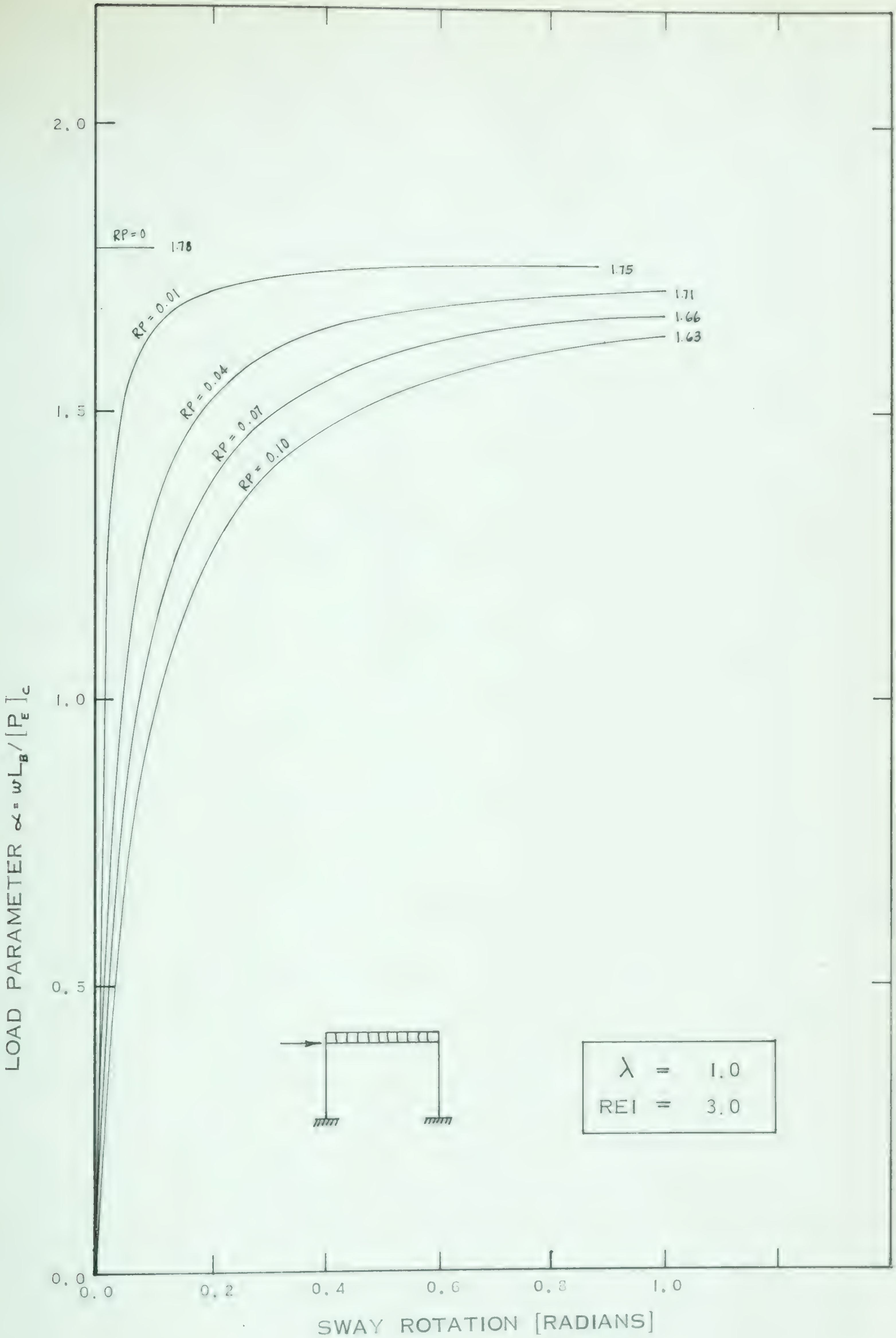


FIGURE 4 – 31 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

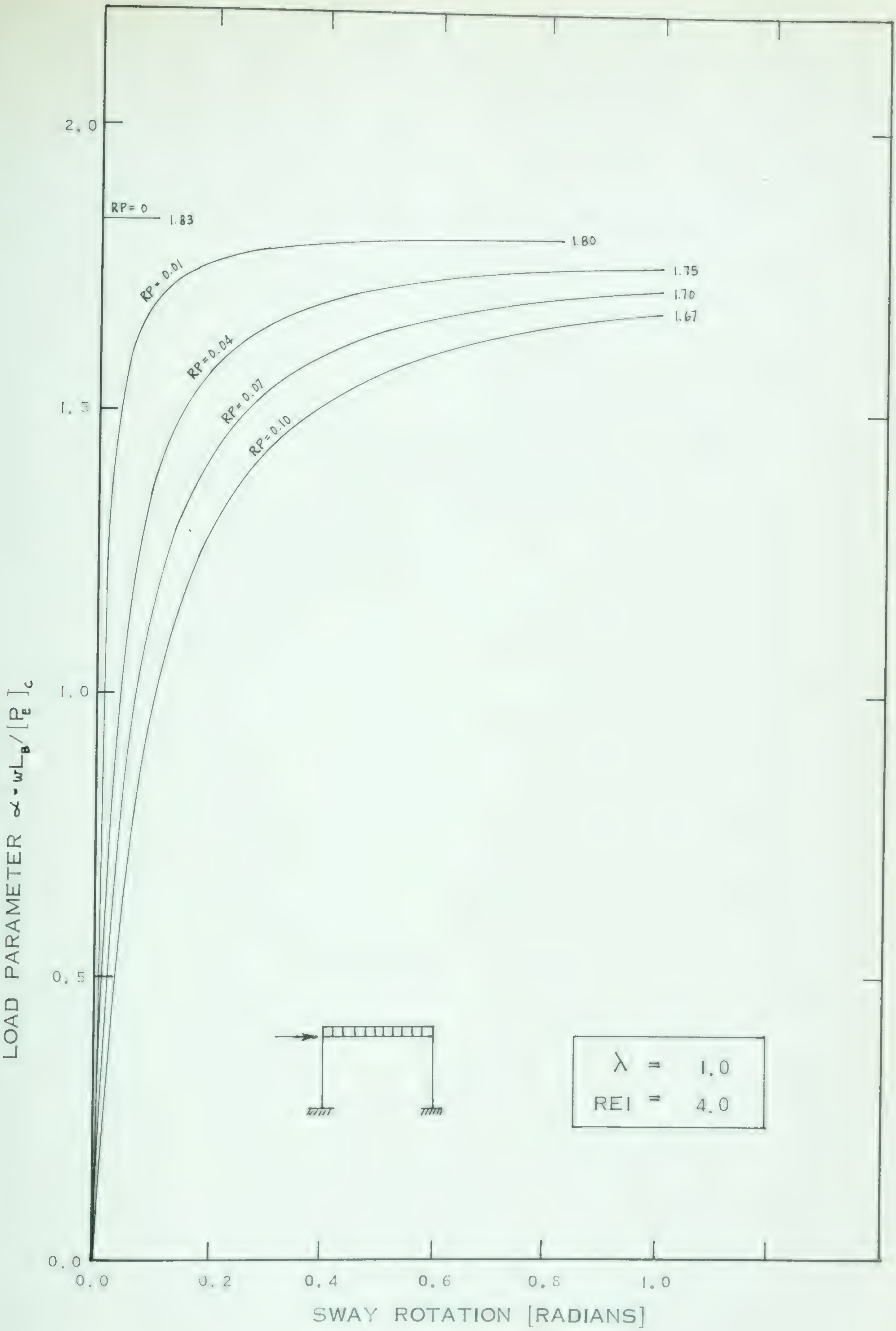


FIGURE 4 – 32 LOAD ~ DEFLECTION – SMALL DEFLECTION THEORY

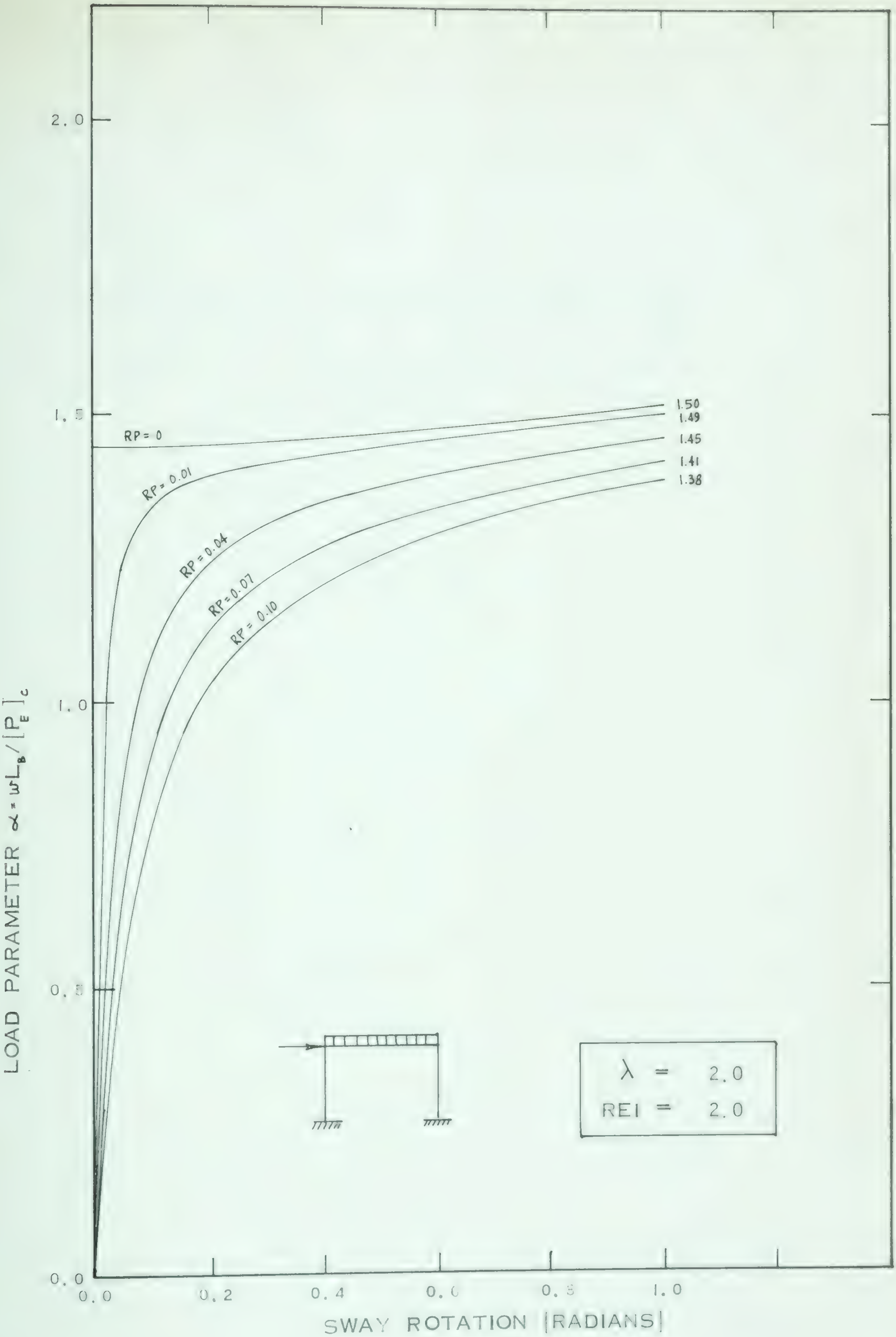


FIGURE 4 – 33 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

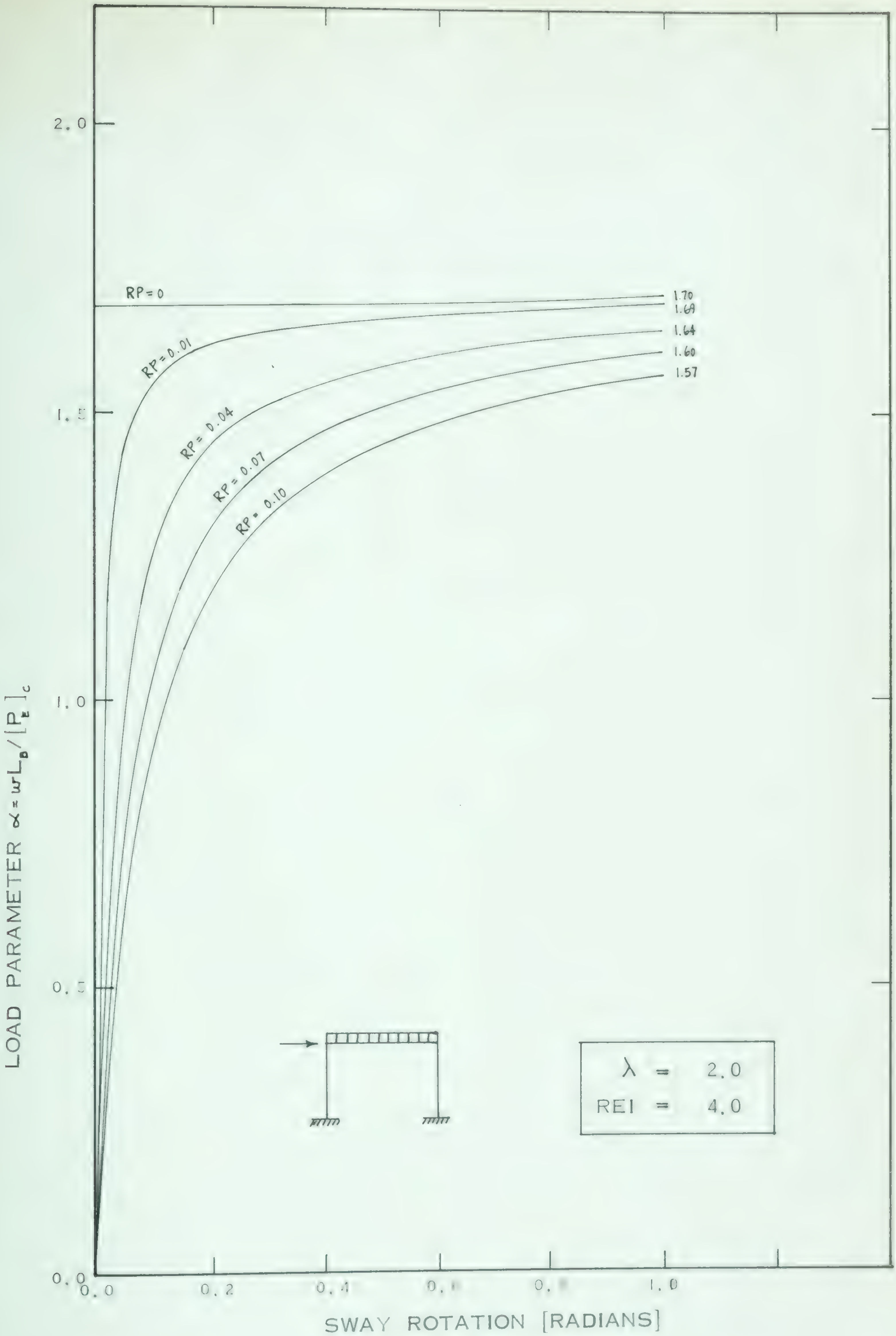


FIGURE 4 – 34 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

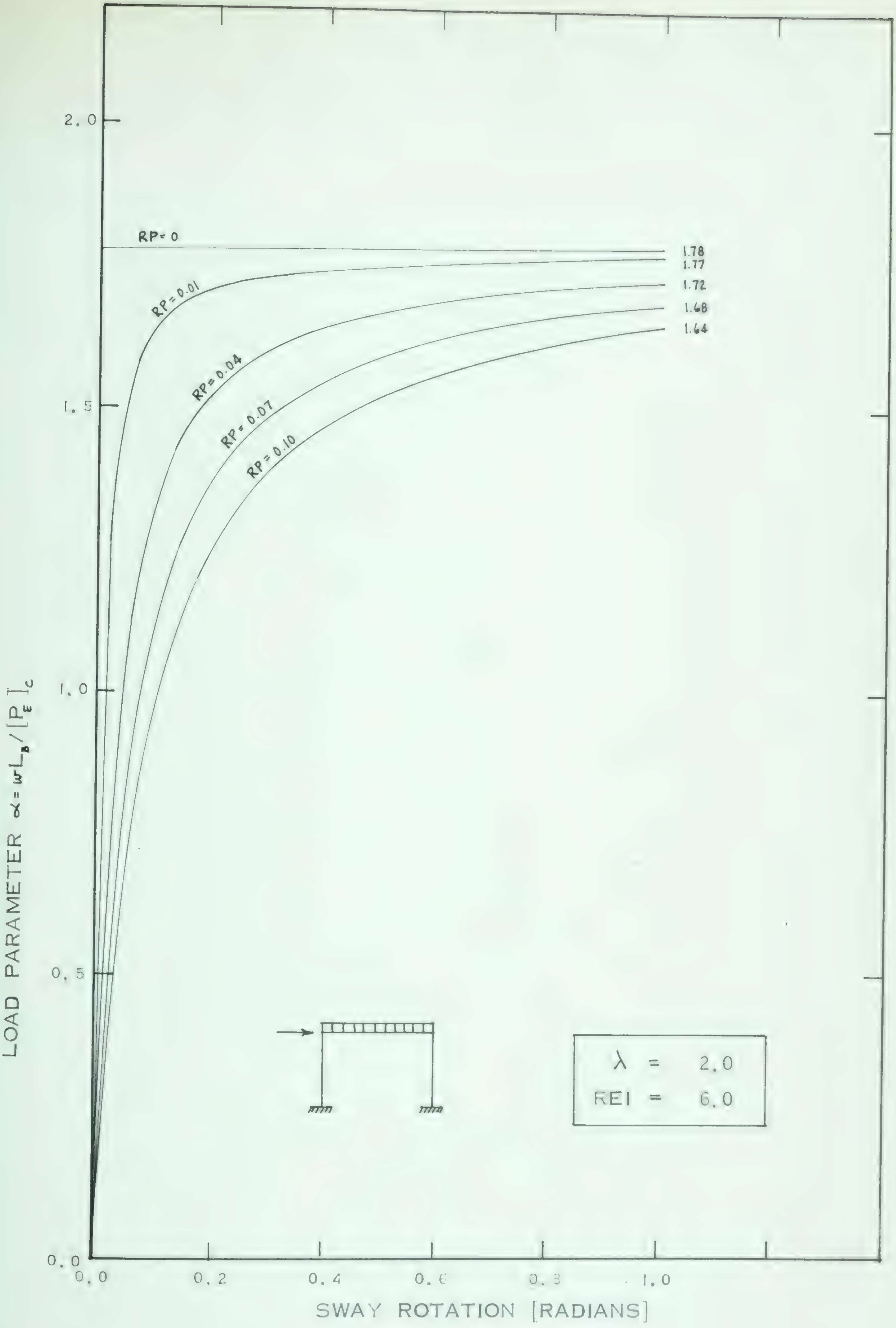


FIGURE 4 – 35 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

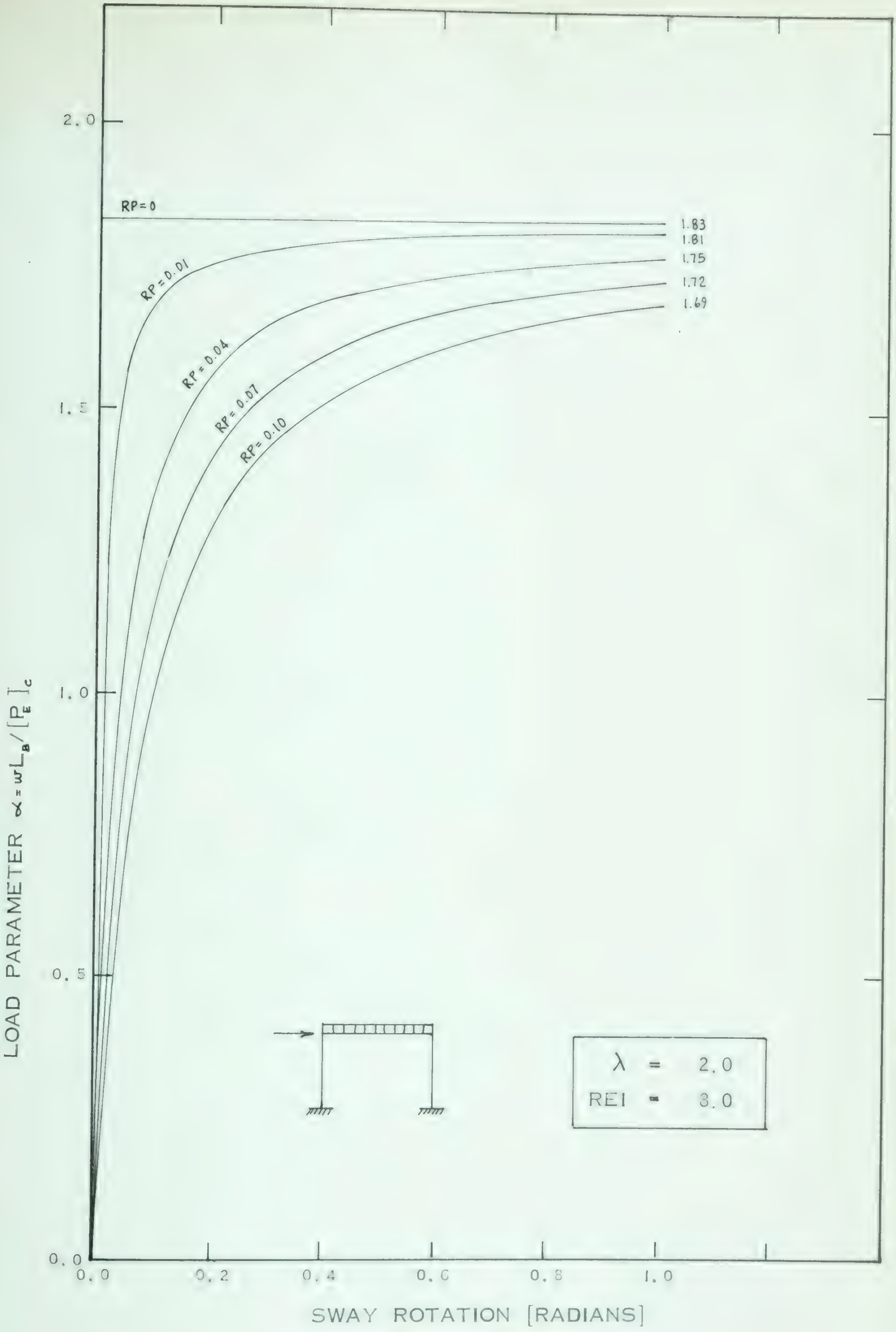


FIGURE 4 – 36 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

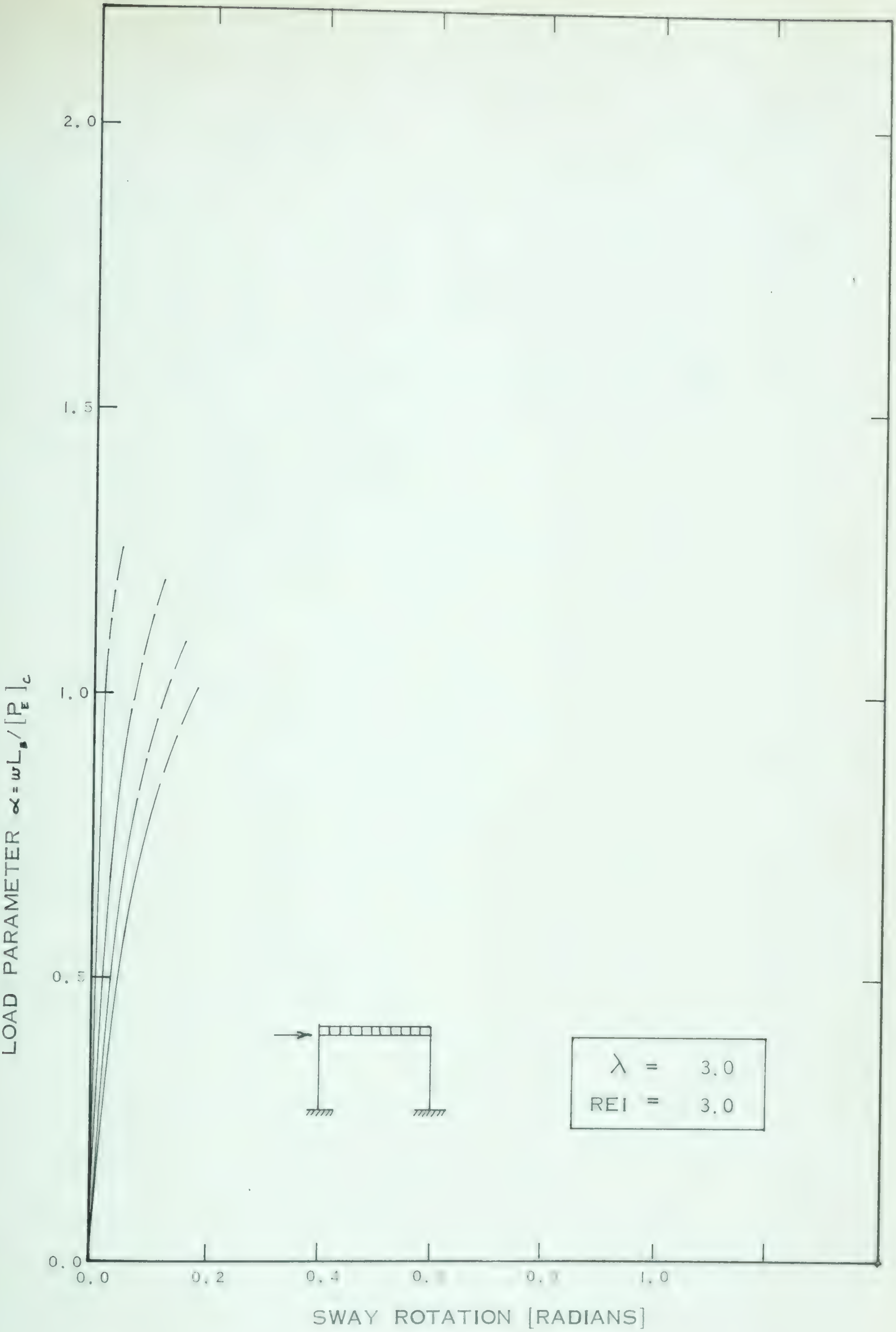


FIGURE 4 – 37 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

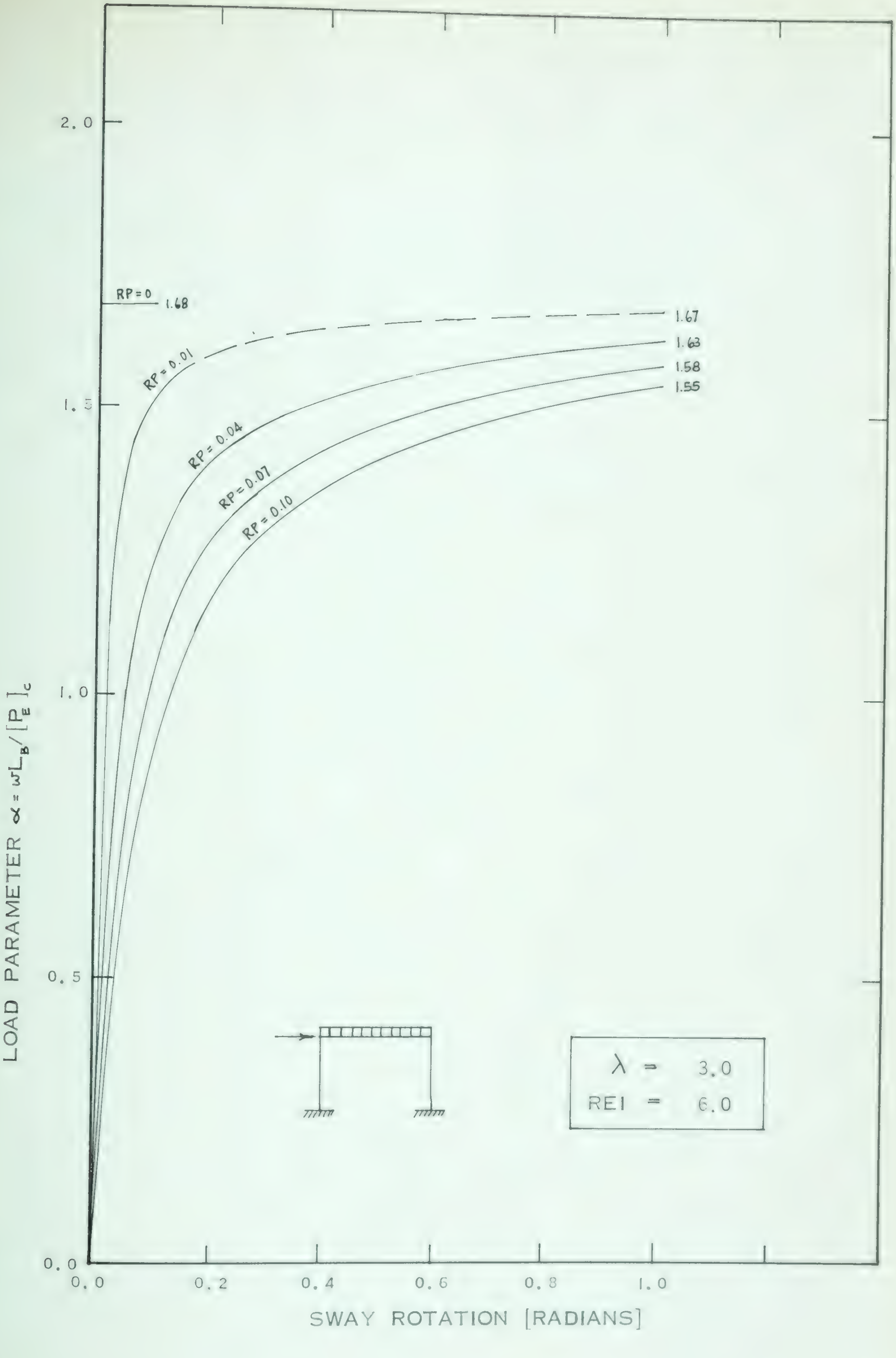


FIGURE 4 – 38 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

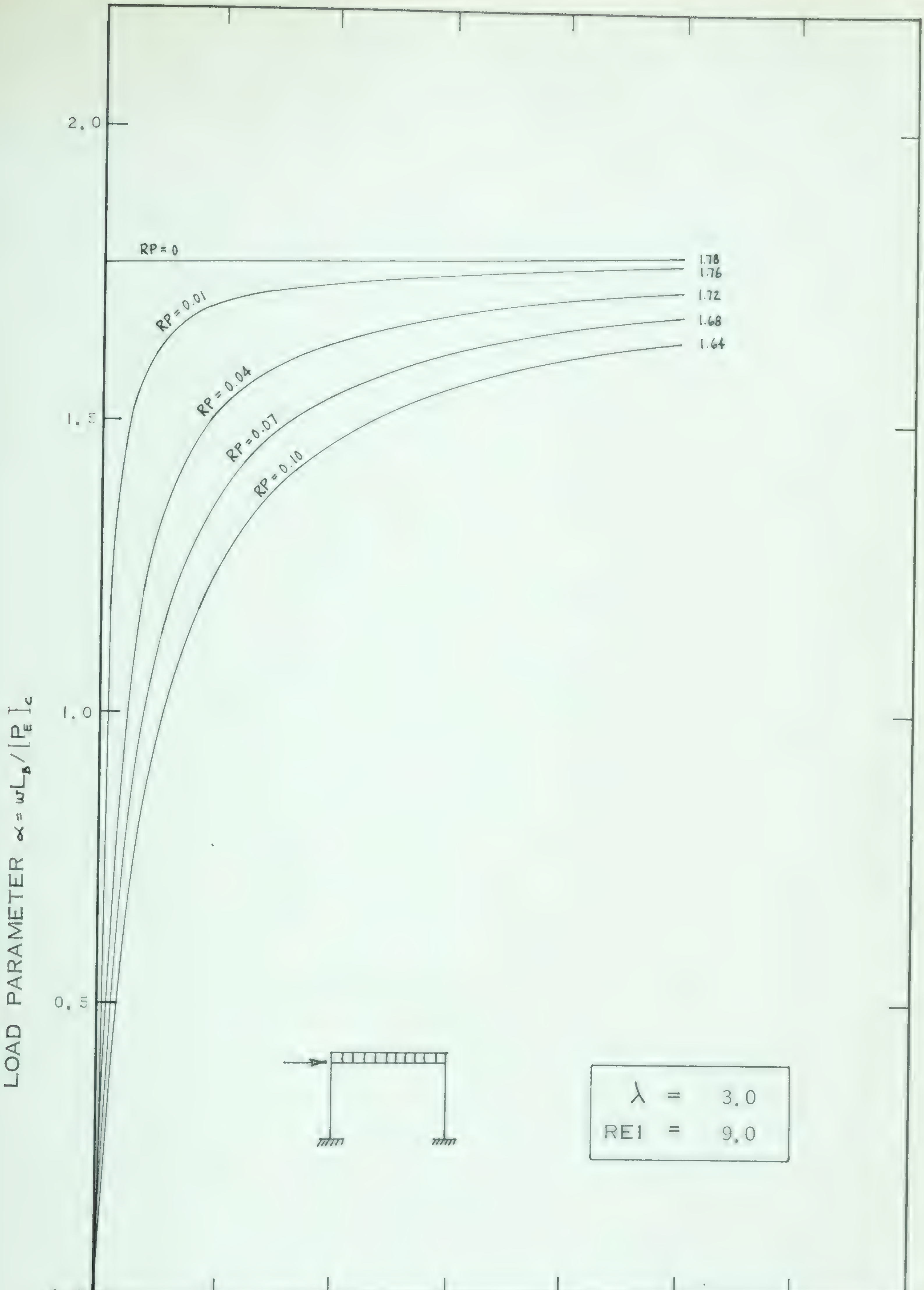


FIGURE 4 – 39 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

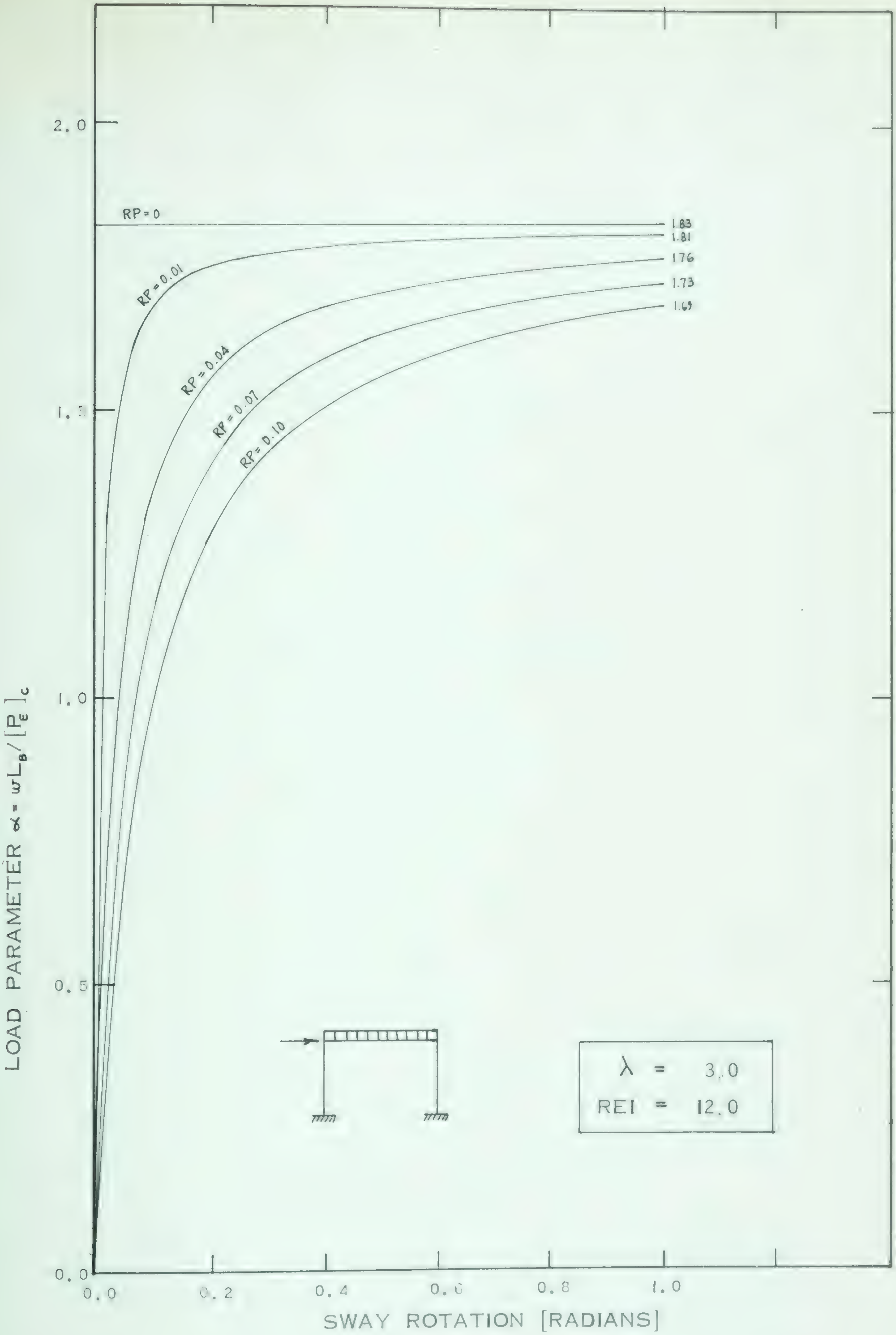


FIGURE 4 – 40 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

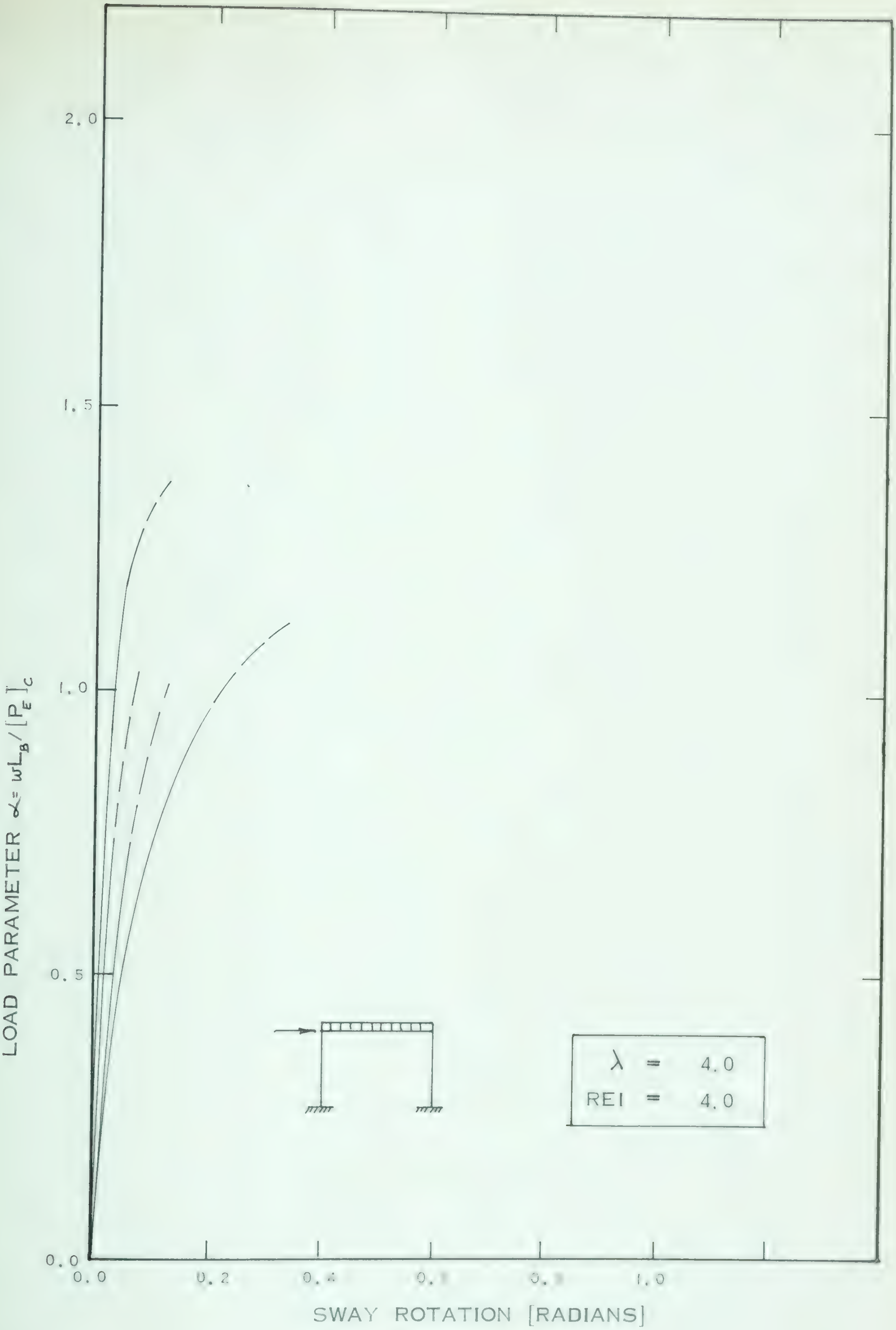


FIGURE 4-41 LOAD vs. DEFLECTION - SMALL DEFLECTION THEORY

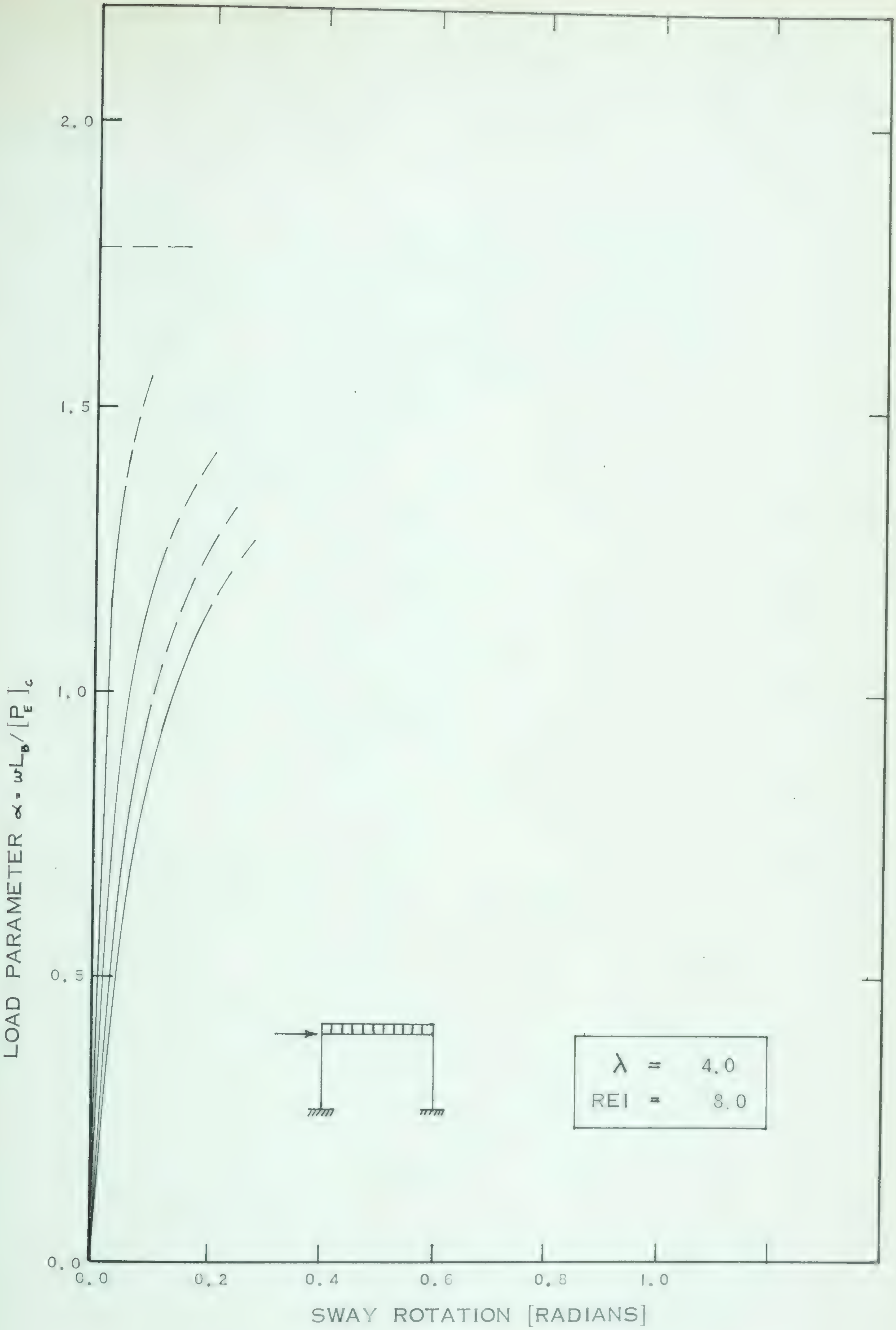


FIGURE 4 – 42 LOAD vs. DEFLECTION – SMALL DEFLECTION THEOR

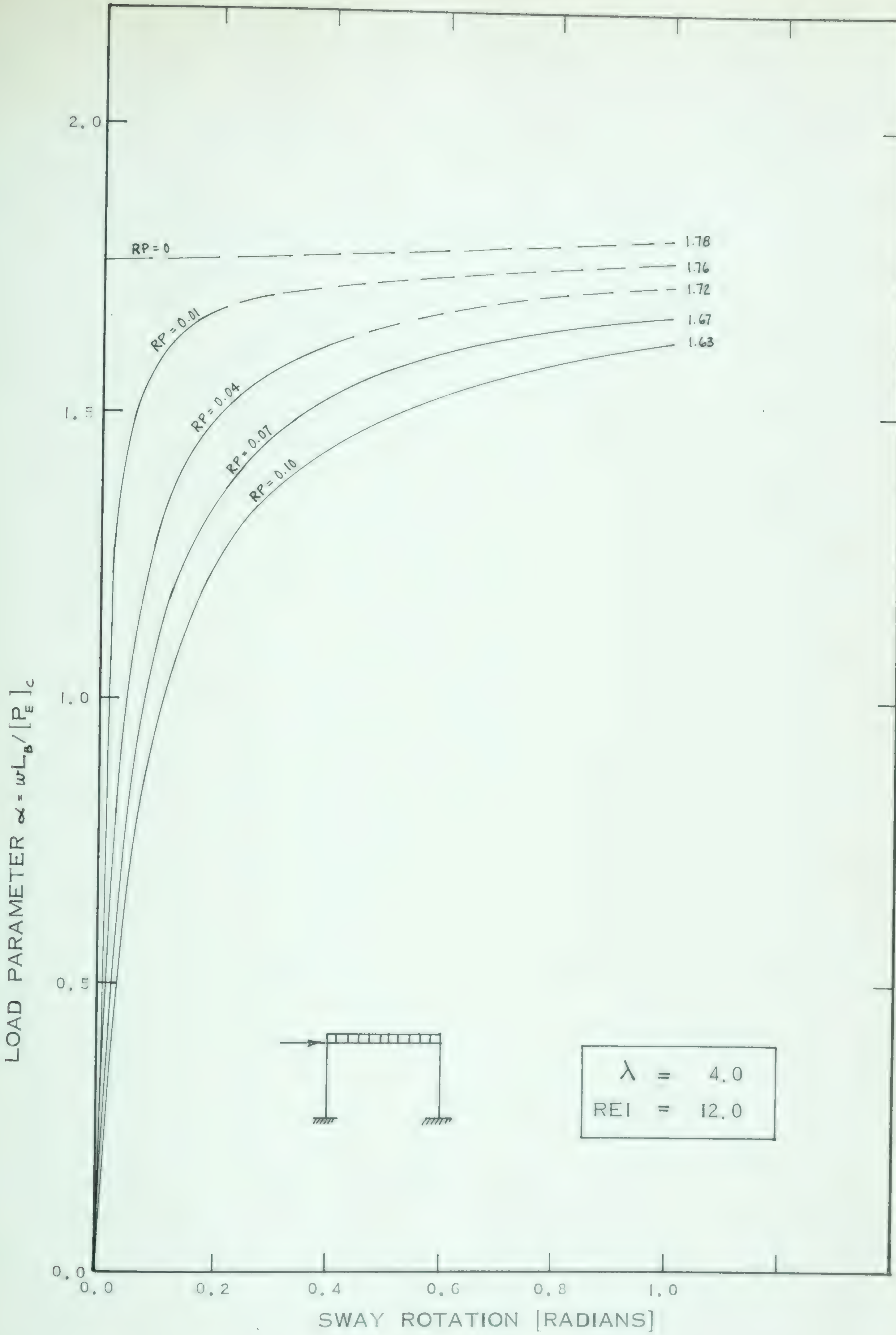


FIGURE 4 – 43 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

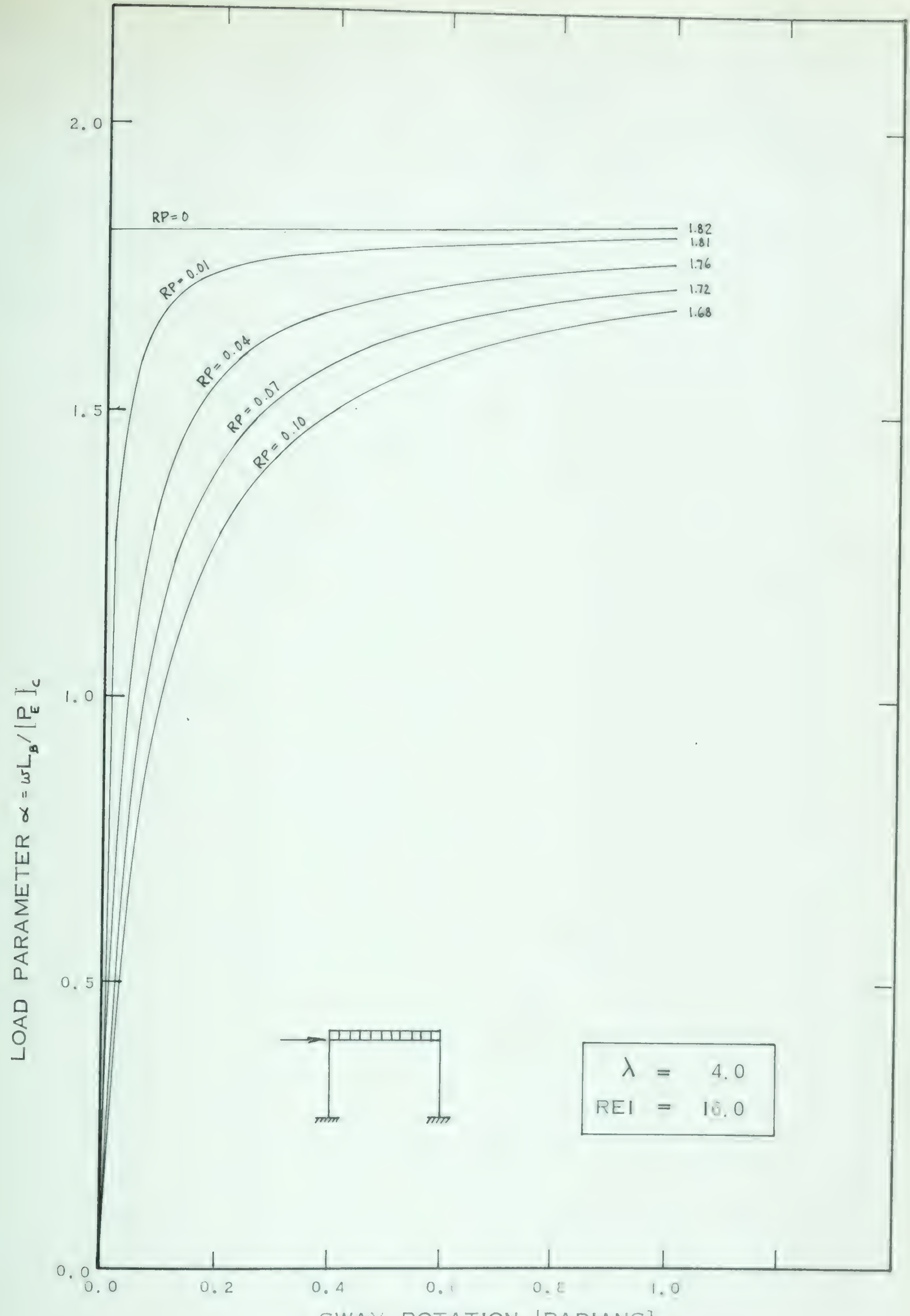


FIGURE 4 – 44 LOAD vs. DEFLECTION – SMALL DEFLECTION THEORY

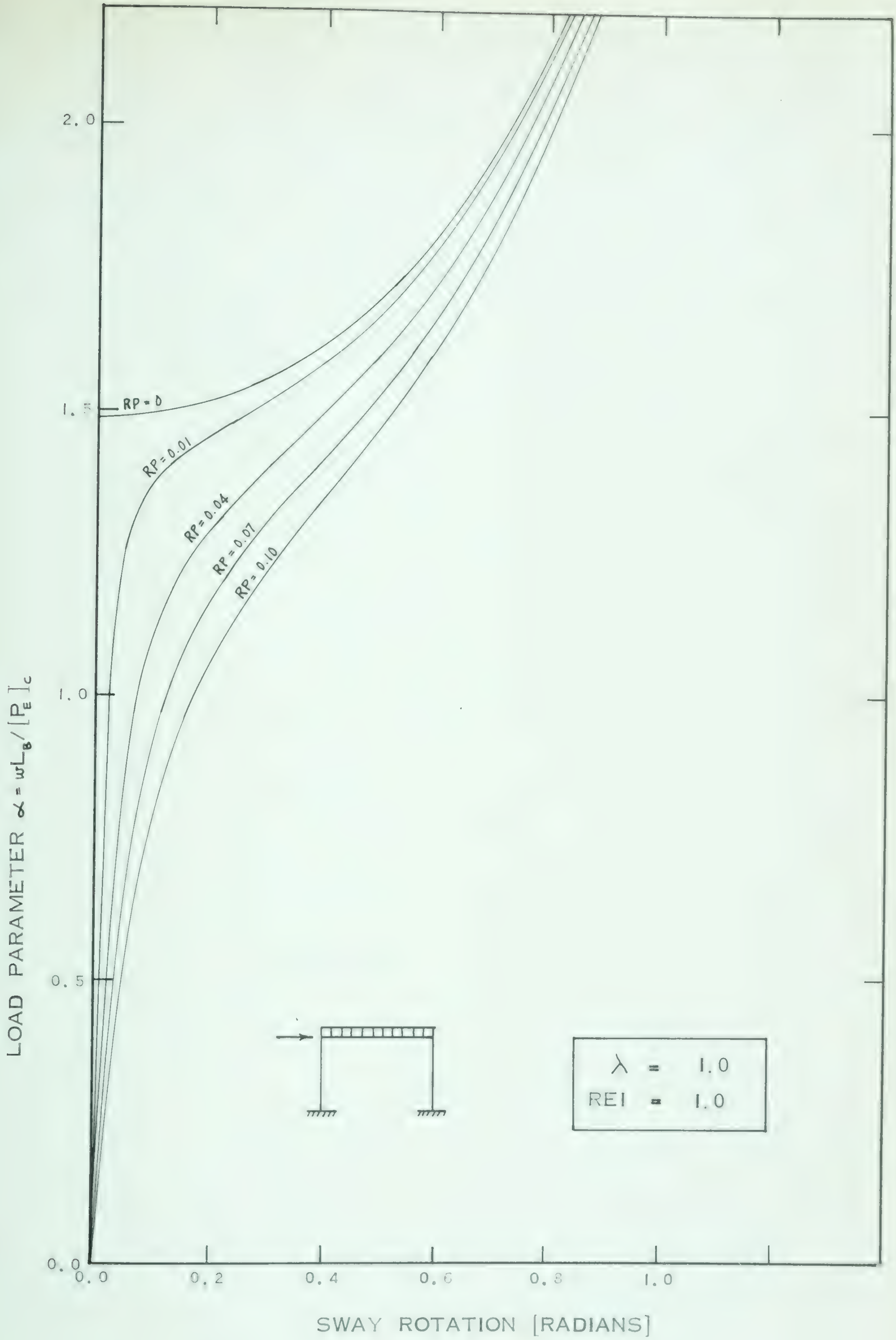


FIGURE 4 – 45 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

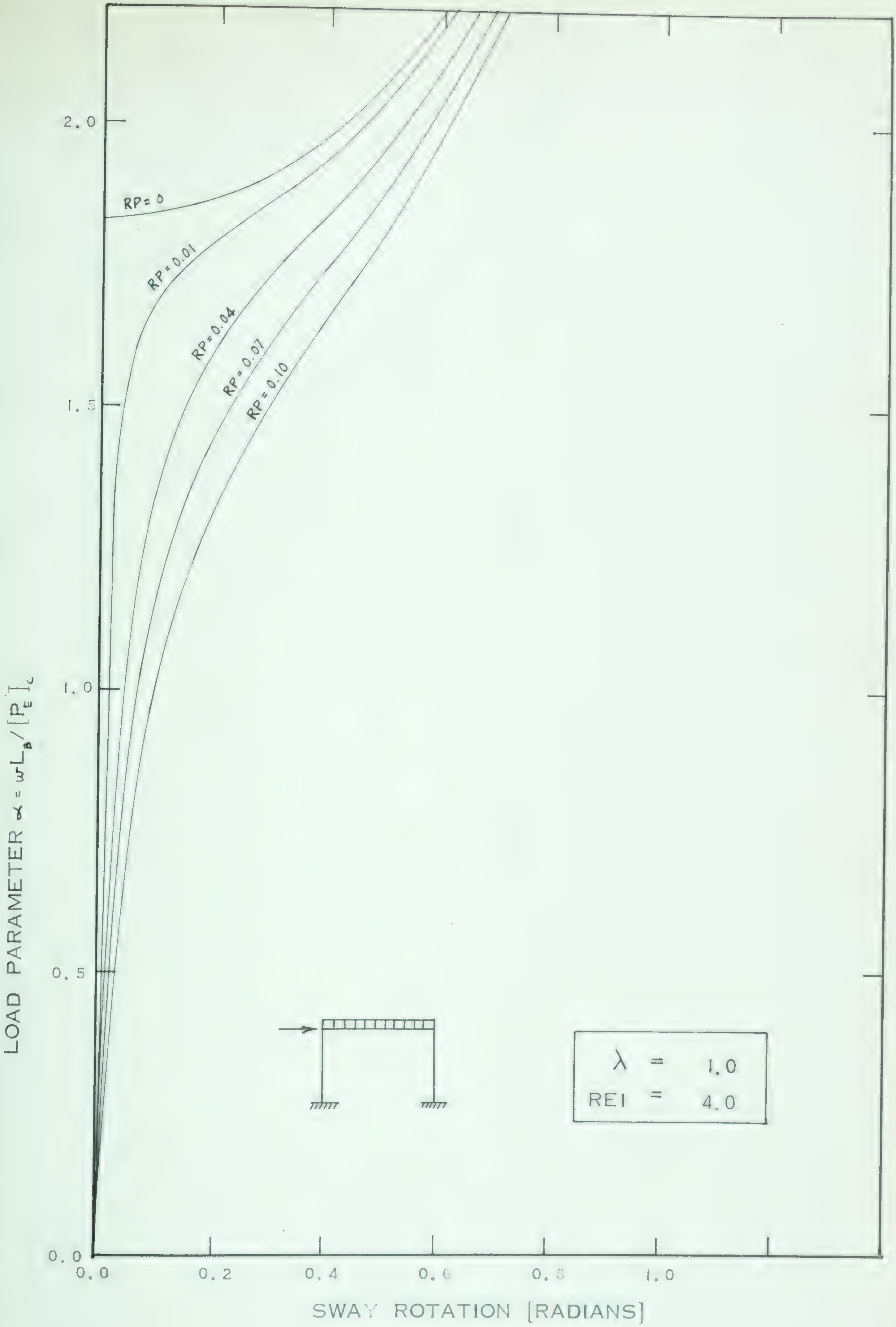


FIGURE 4 – 46 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

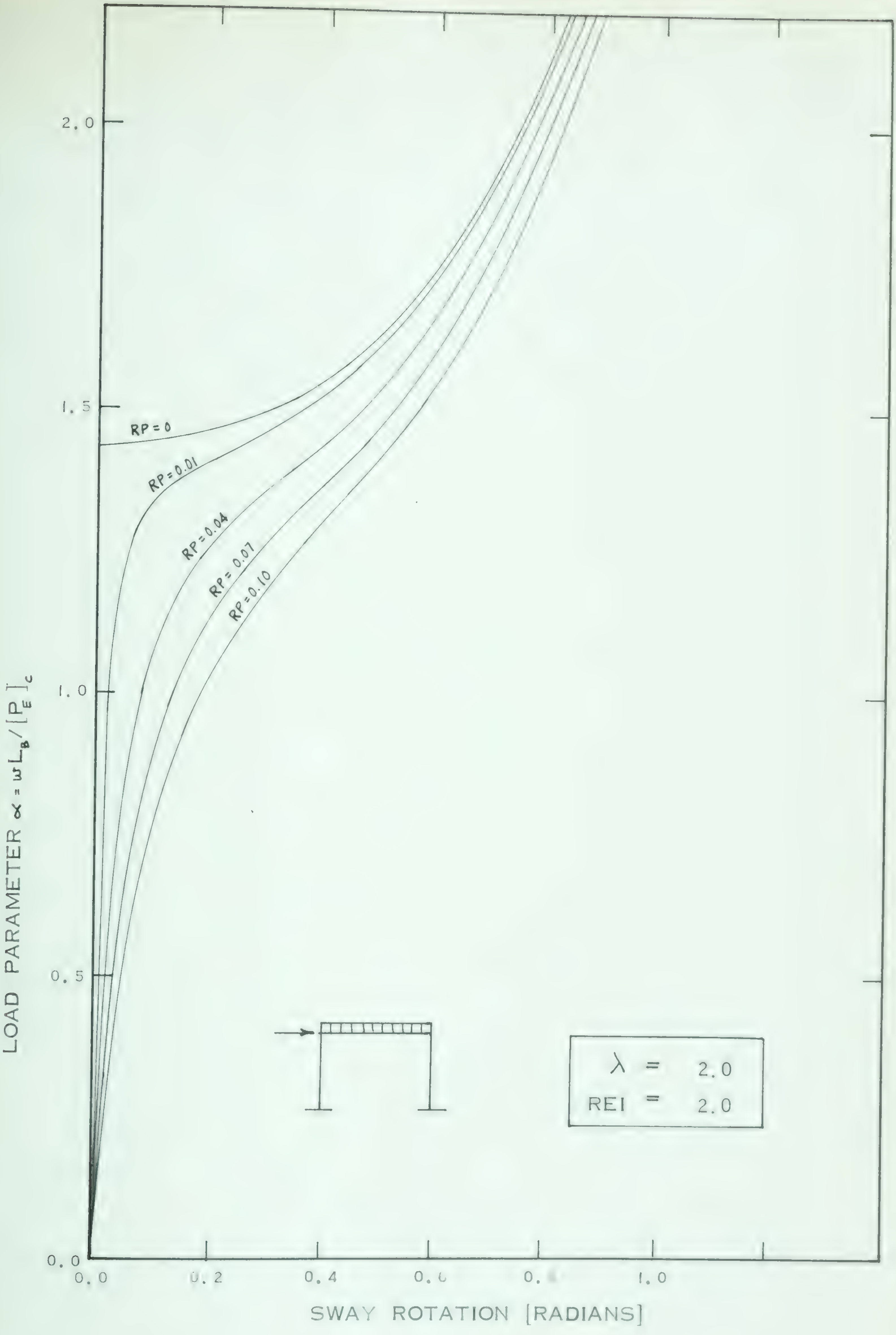


FIGURE 4 – 47 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

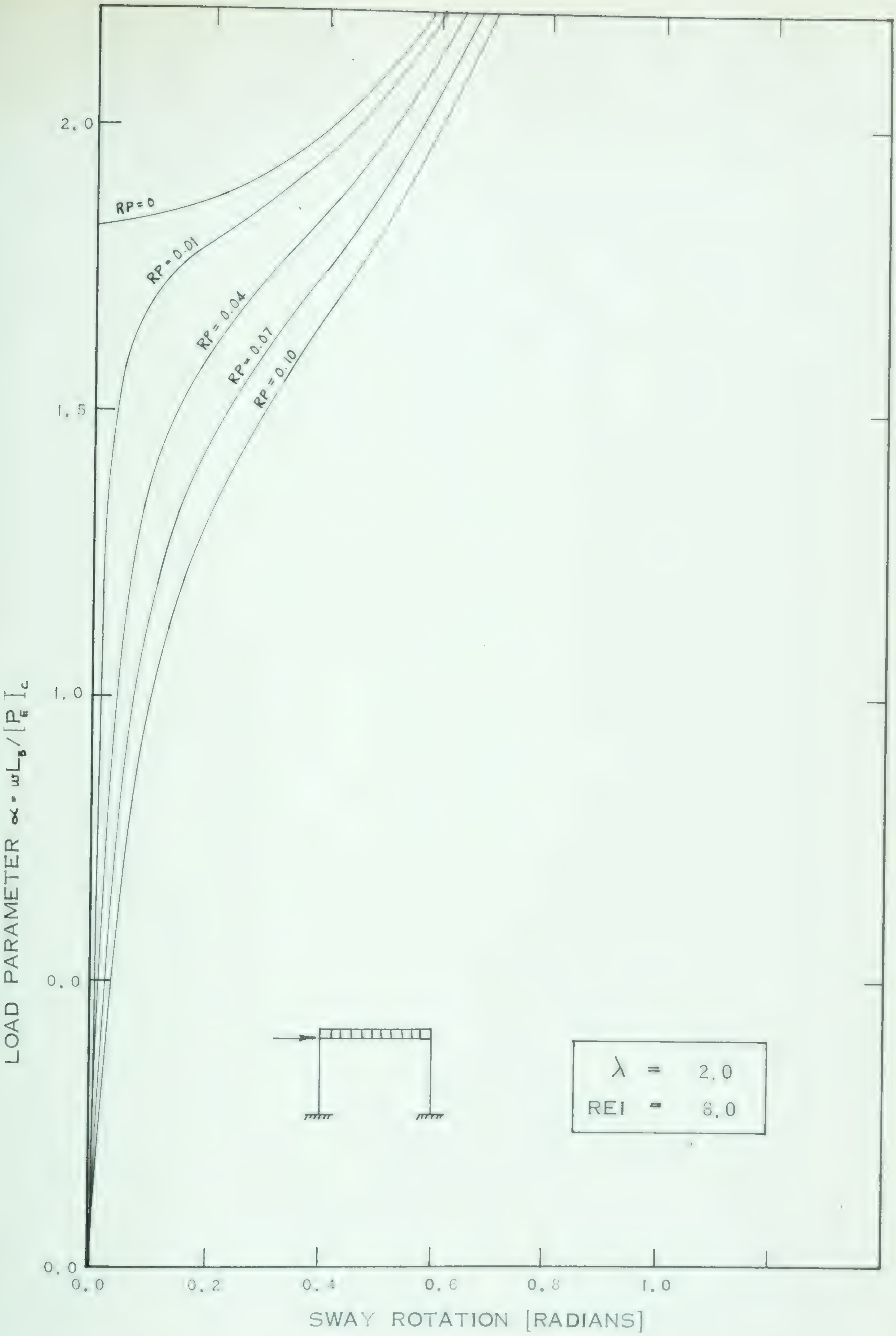


FIGURE 4 – 48 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

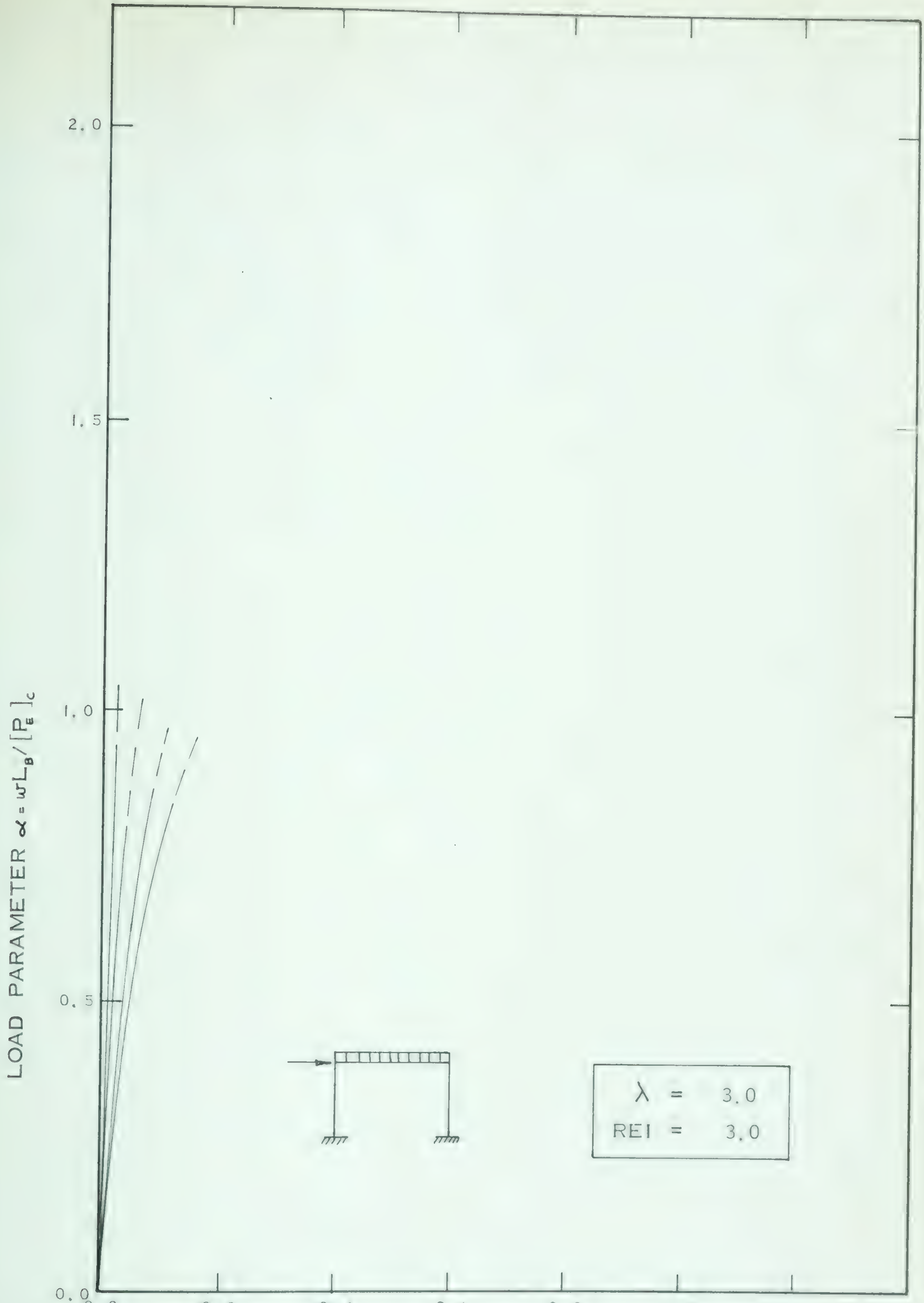


FIGURE 4 – 49 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

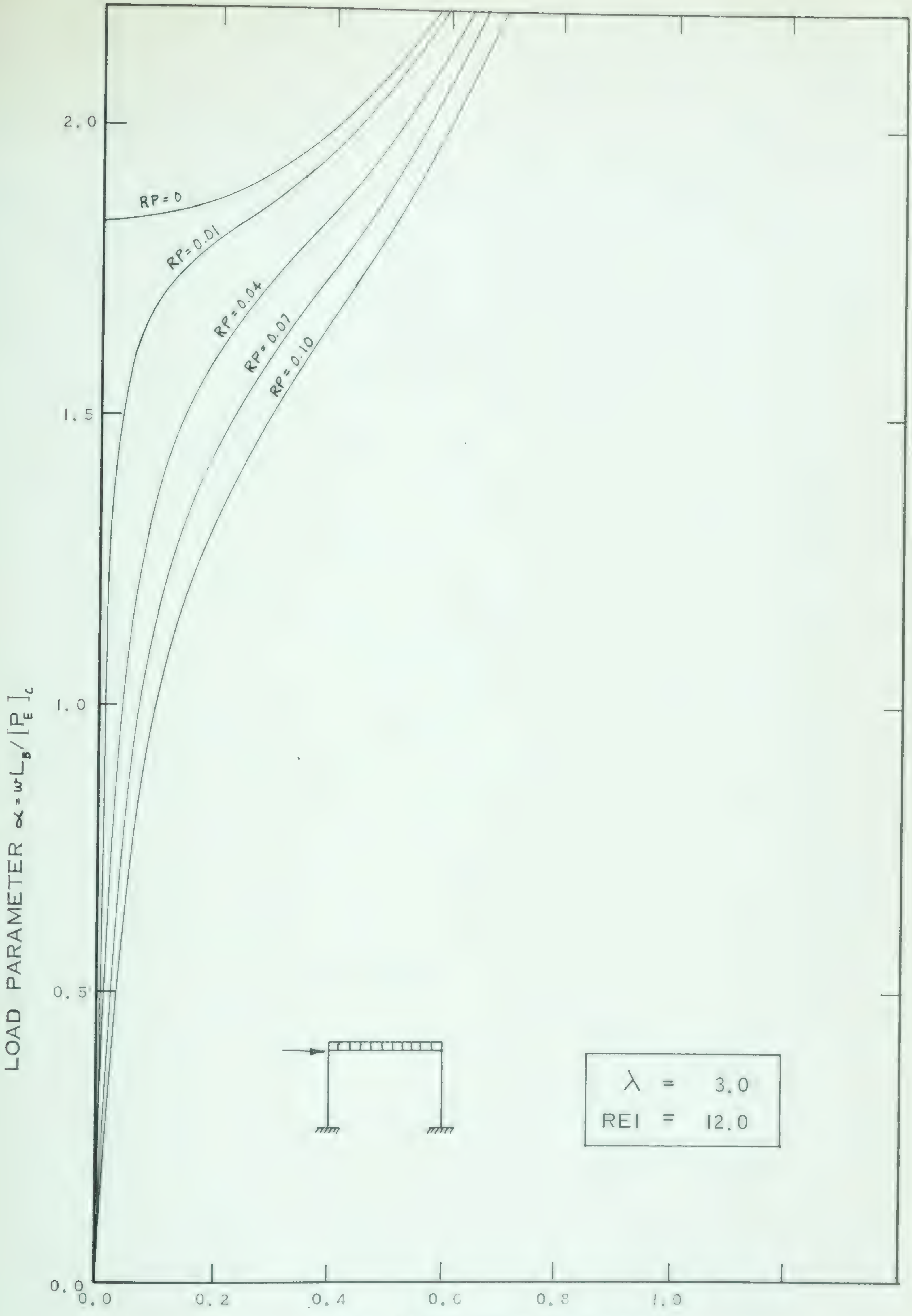


FIGURE 4 – 50 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

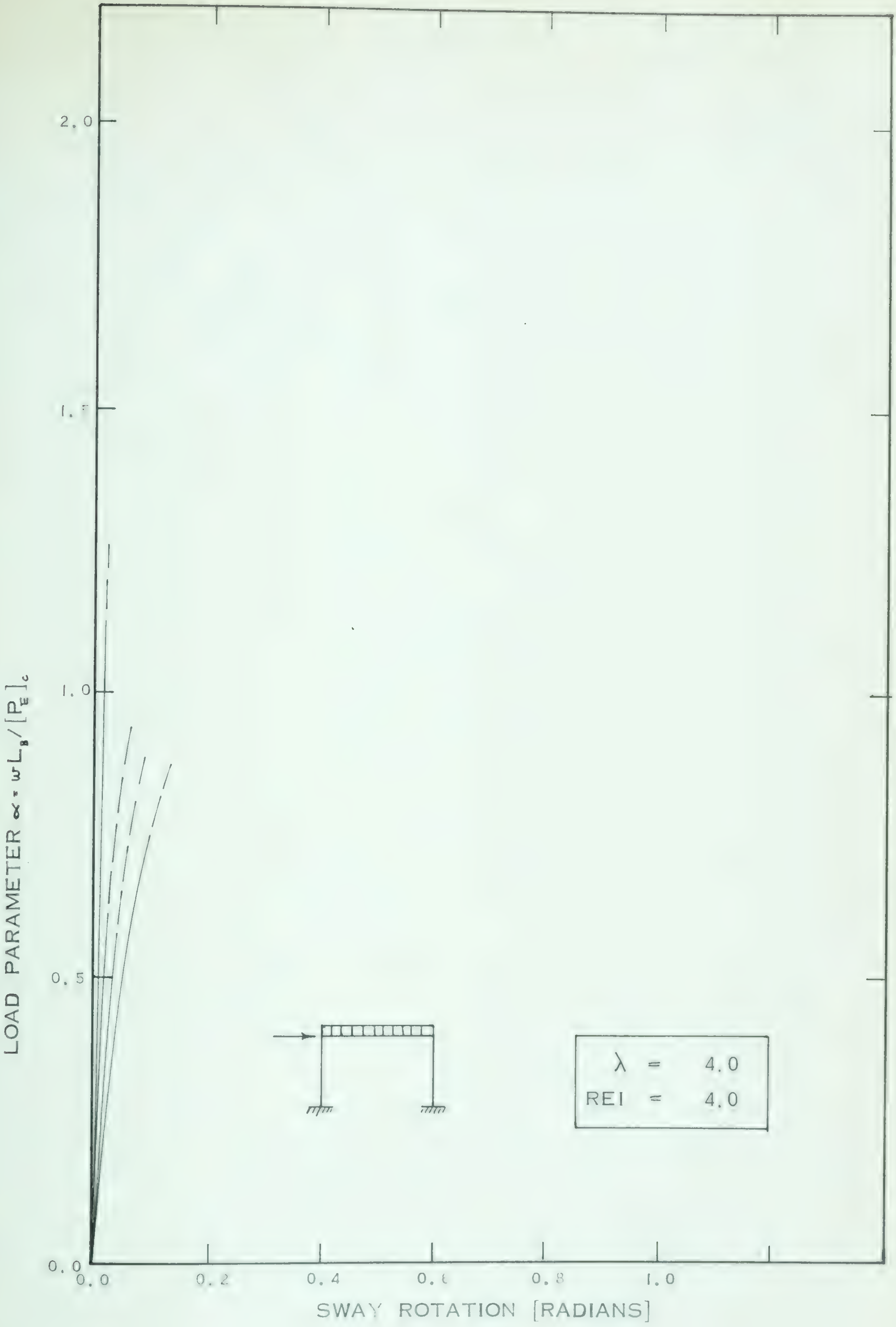


FIGURE 4 – 51 LOAD . DEFLECTION LARGE DEFLECTION THEORY

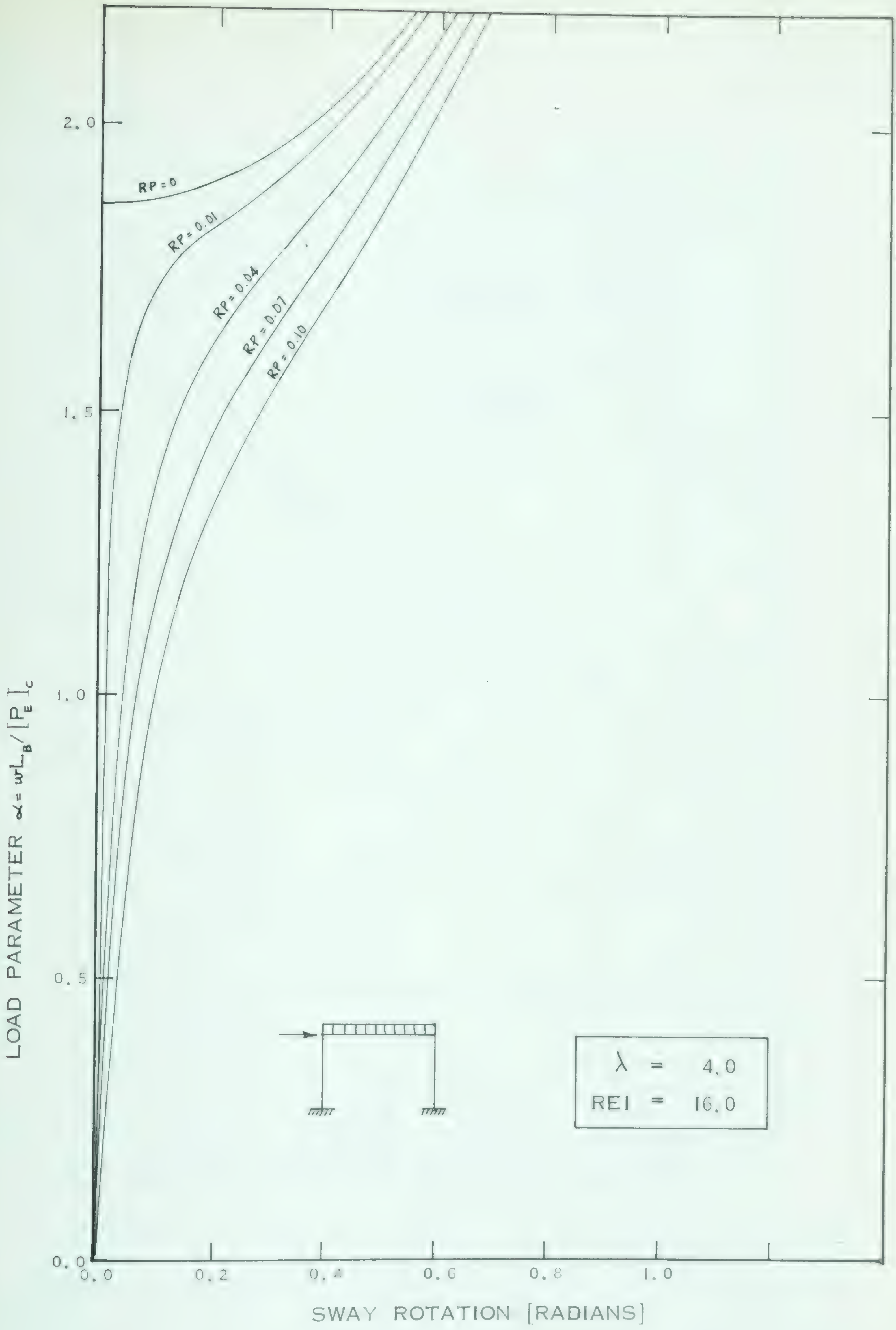
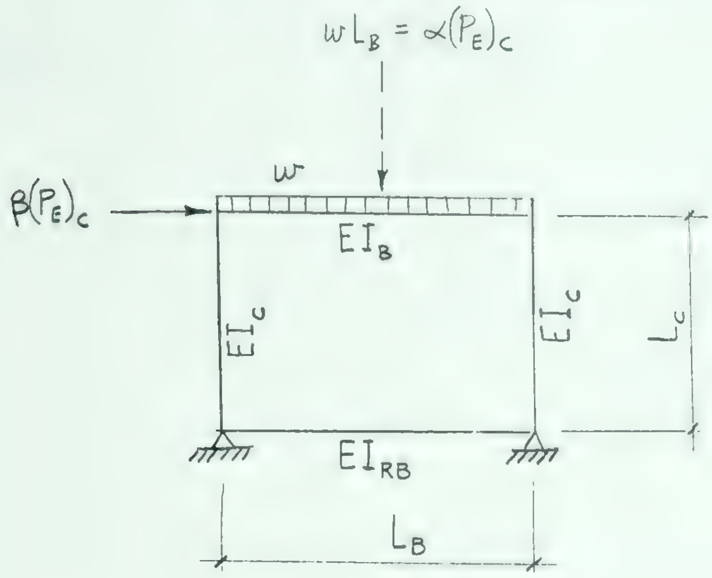


FIGURE 4 – 52 LOAD vs. DEFLECTION – LARGE DEFLECTION THEORY

λ ↓	REI →	1.0	2.0	3.0	4.0	6.0	8.0	9.0	12.0	16.0
1.0	RP=0	1.51	1.69	1.78	1.83					
	RP=0.01	1.50	1.68	1.75	1.80					
	RP=0.04	1.46	1.63	1.71	1.75					
	RP=0.07	1.42	1.59	1.66	1.70					
	RP=0.10	1.39	1.55	1.63	1.67					
2.0	RP=0		1.50		1.70	1.78	1.83			
	RP=0.01		1.49		1.69	1.77	1.81			
	RP=0.04		1.45		1.64	1.72	1.75			
	RP=0.07		1.41		1.60	1.68	1.72			
	RP=0.10		1.38		1.57	1.64	1.69			
3.0	RP=0					1.68		1.78	1.83	
	RP=0.01					1.67		1.76	1.81	
	RP=0.04					1.63		1.72	1.76	
	RP=0.07					1.58		1.68	1.73	
	RP=0.10					1.55		1.64	1.69	
4.0	RP=0								1.78	1.82
	RP=0.01								1.76	1.81
	RP=0.04								1.72	1.76
	RP=0.07								1.67	1.72
	RP=0.10								1.63	1.68

Table of Elastic Critical Loads For Frame With Fixed Bases

4-4 FRAME WITH PARTIAL BASE FIXITY



$RP = \beta/\alpha$

$\lambda = L_B/L_c$

$REI = EI_B/EI_c$

$\eta = (EI/L)_{RB}/(EI/L)_B = EI_{RB}/EI_B$

FIGURE 4-53

One more variable is introduced for the frame with partial base fixity compared with the previous two cases. The relationships between the elastic critical loads and the restraint parameter η were plotted and are presented in Figures 4-54 to 4-61 for the two values of horizontal load considered. In plotting these graphs, the critical load used was the greatest value of the load parameter α obtained for a range of sway rotations less than or equal to 1.0.

When $\eta = 0$, the elastic critical load is the same as for the hinged base frame. As η increases above zero, the elastic critical load increases rapidly at first, but levels off to approach asymptotically the value of the elastic critical load for the corresponding fixed-base frame as η approaches infinity.

It can be observed for all combinations of variables investigated that only a small amount of base restraint increases the critical load considerably. This behavior is most pronounced when the beam stiffness is large. If the base restraint is equal to the stiffness of the top beam,

the critical load is almost equal to that of the corresponding fixed-base frame.

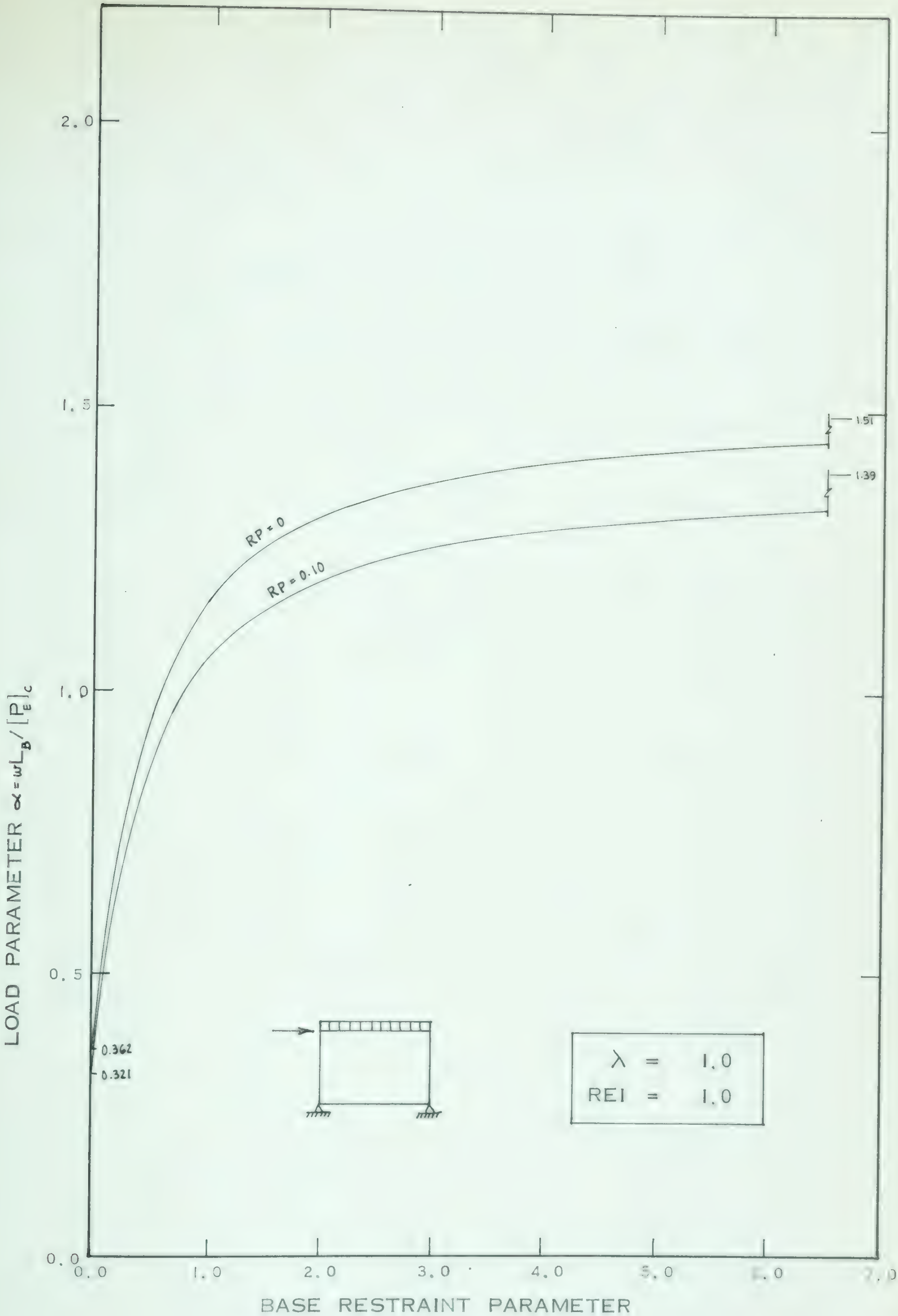


FIGURE 4 – 54 LOAD vs. BASE RESTRAINT

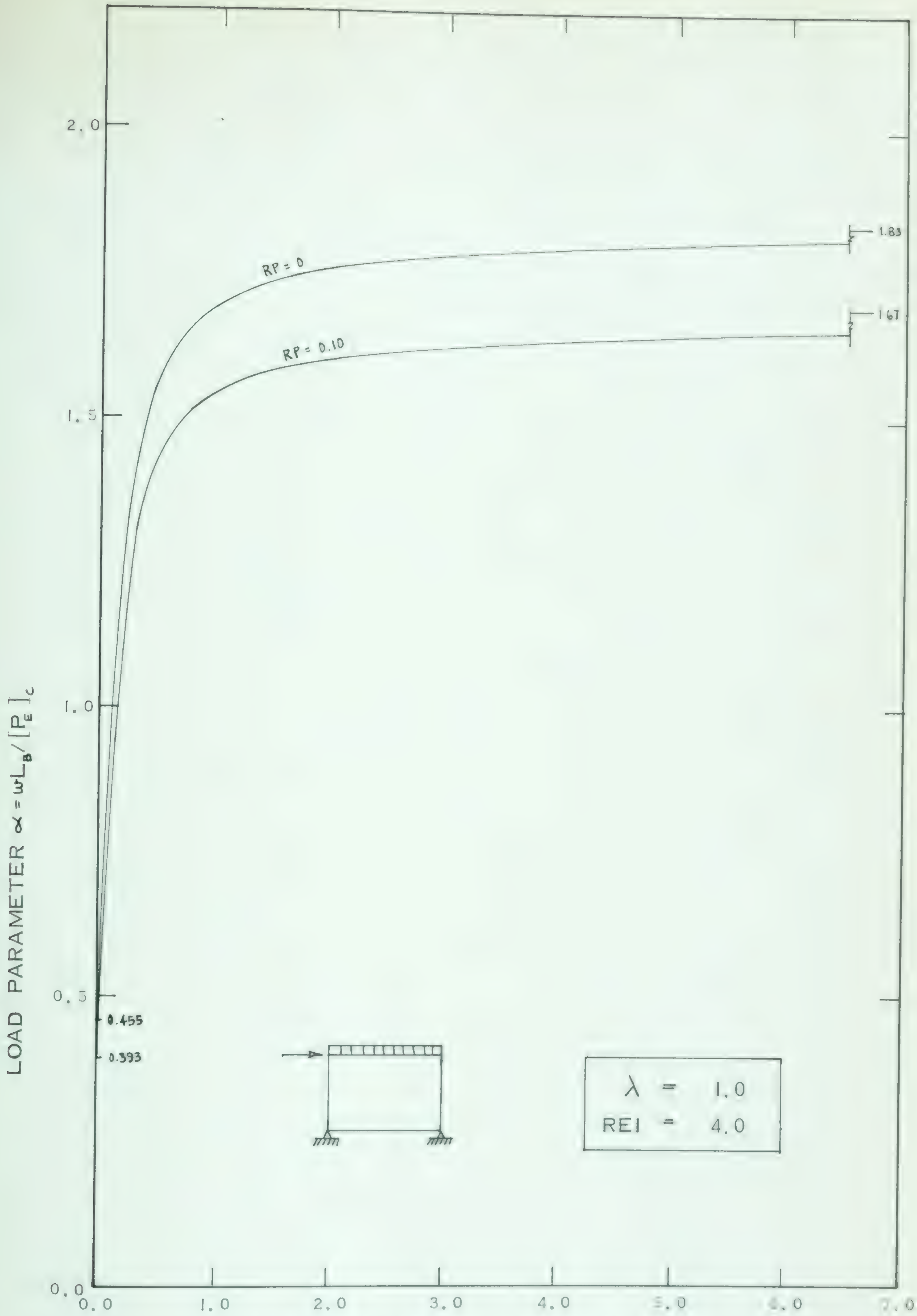


FIGURE 4 – 55 LOAD vs. BASE RESTRAINT

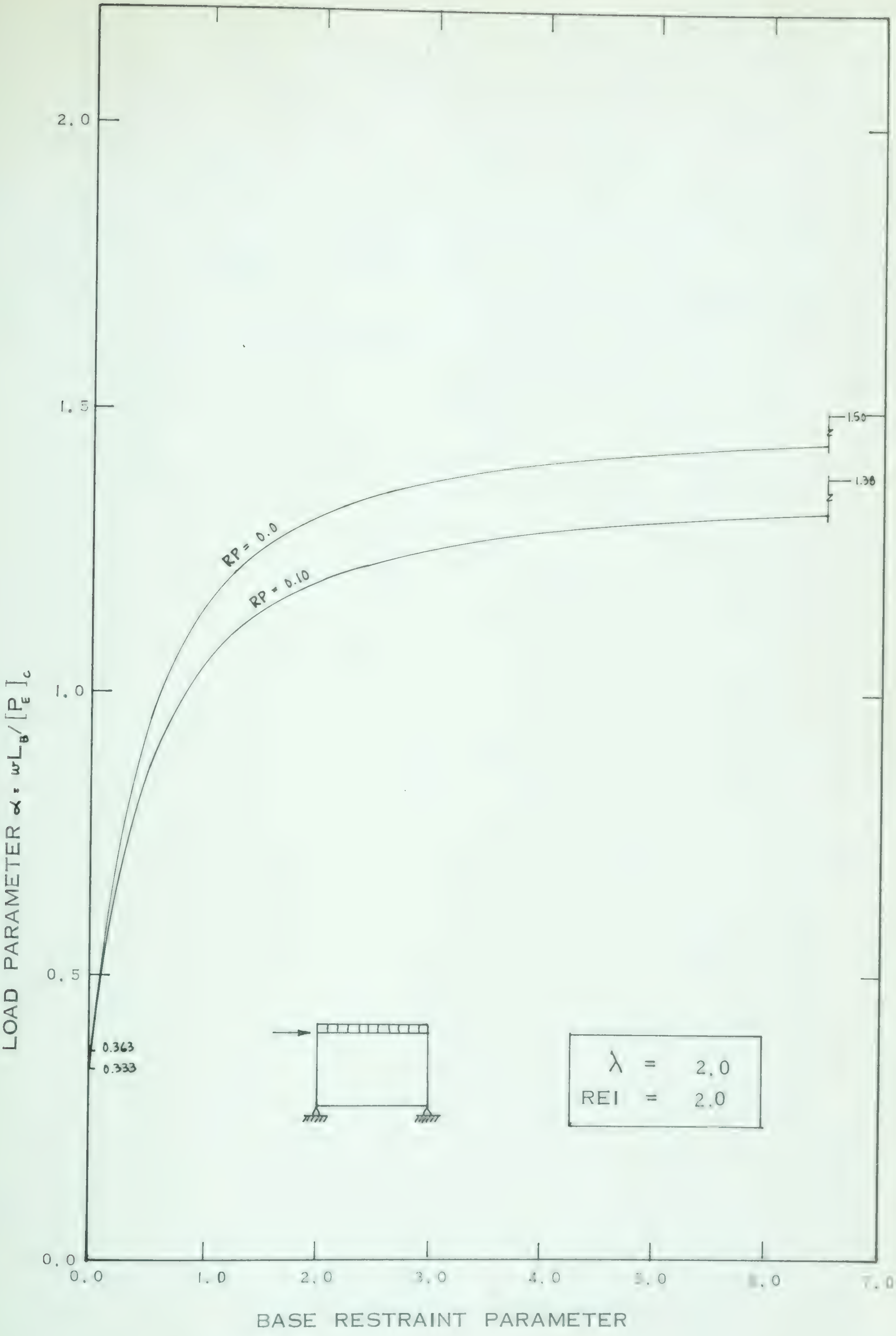


FIGURE 4 – 56 LOAD vs. BASE RESTRAINT

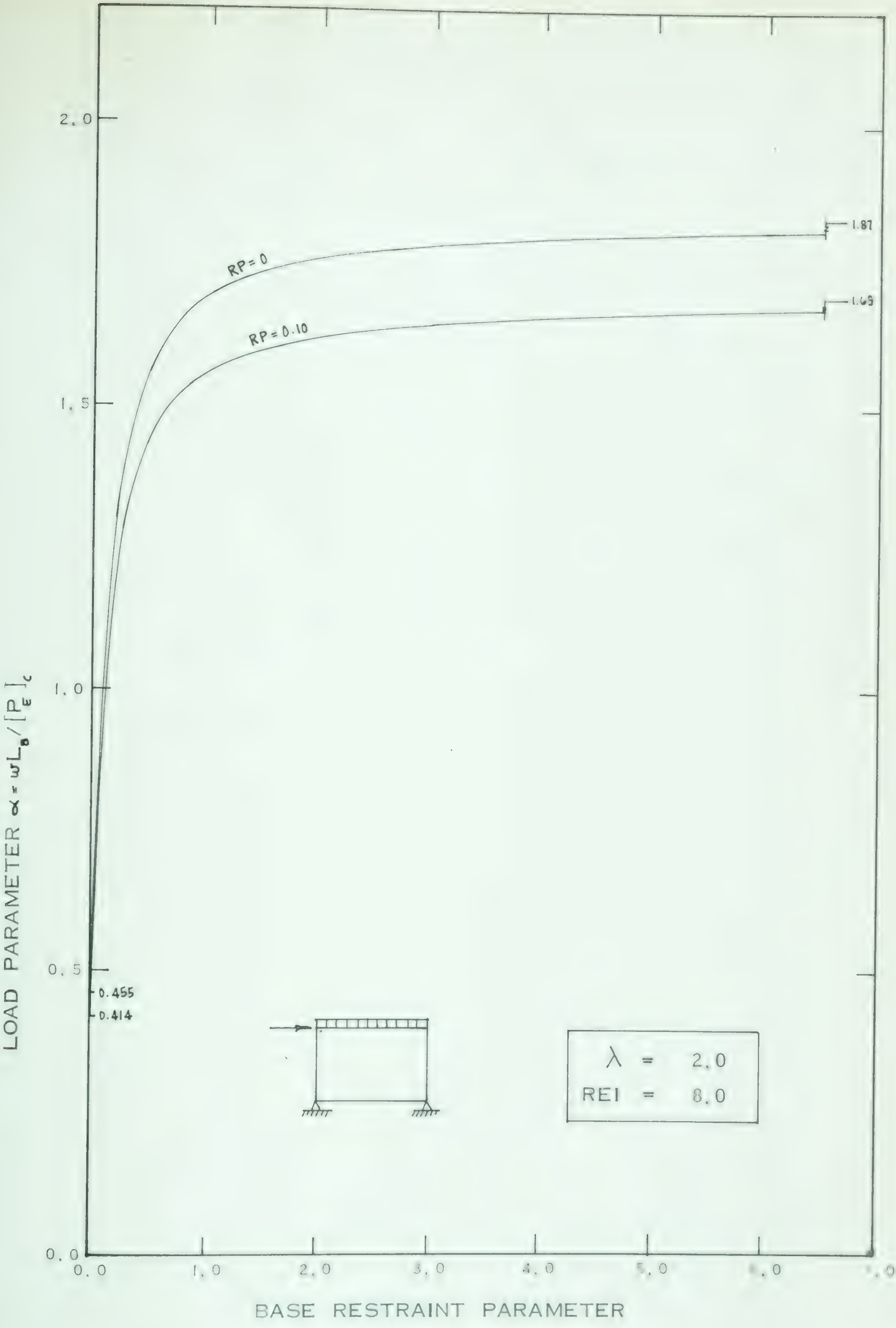


FIGURE 4 – 57 LOAD vs. BASE RESTRAINT

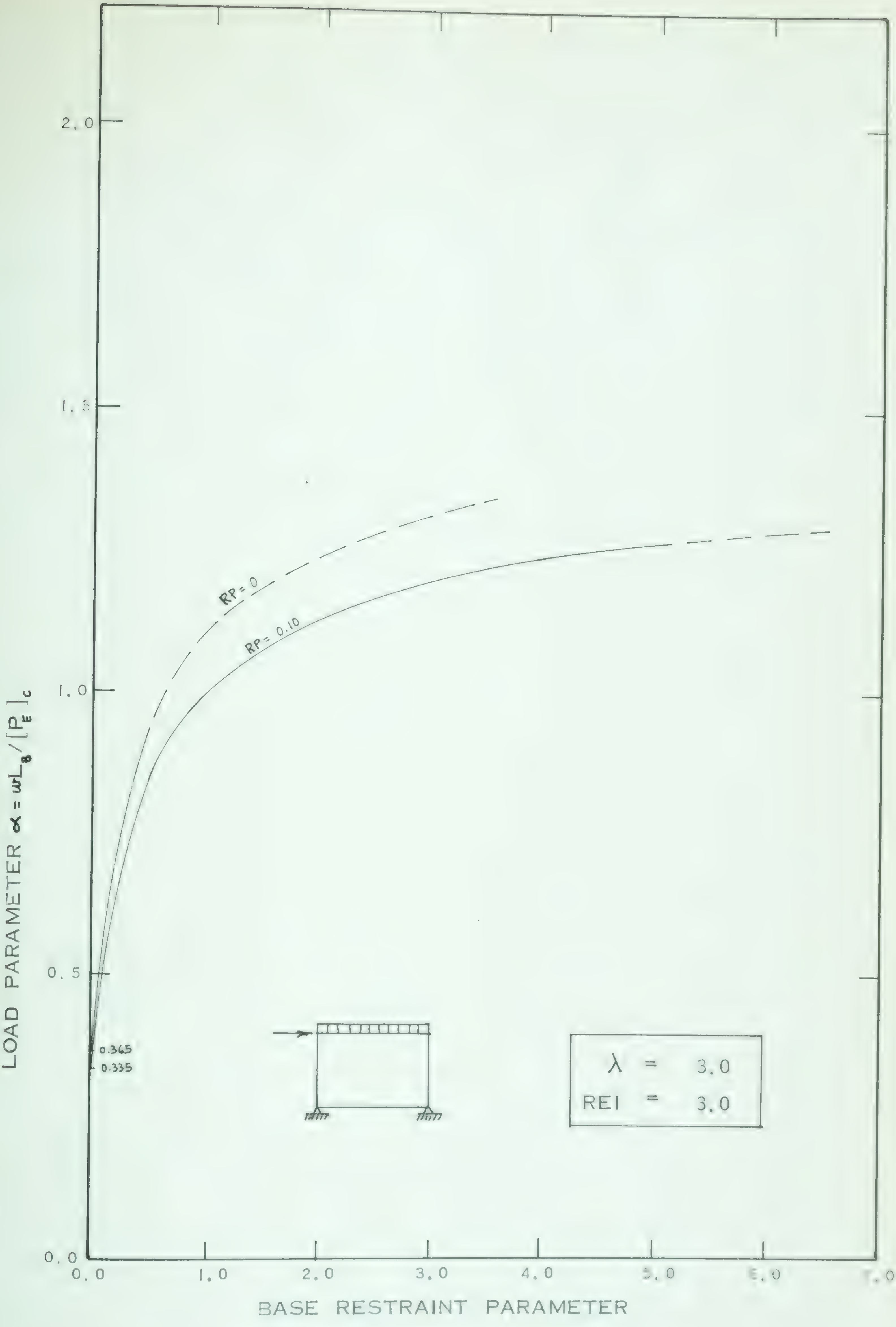


FIGURE 4 – 58 LOAD vs. BASE RESTRAINT

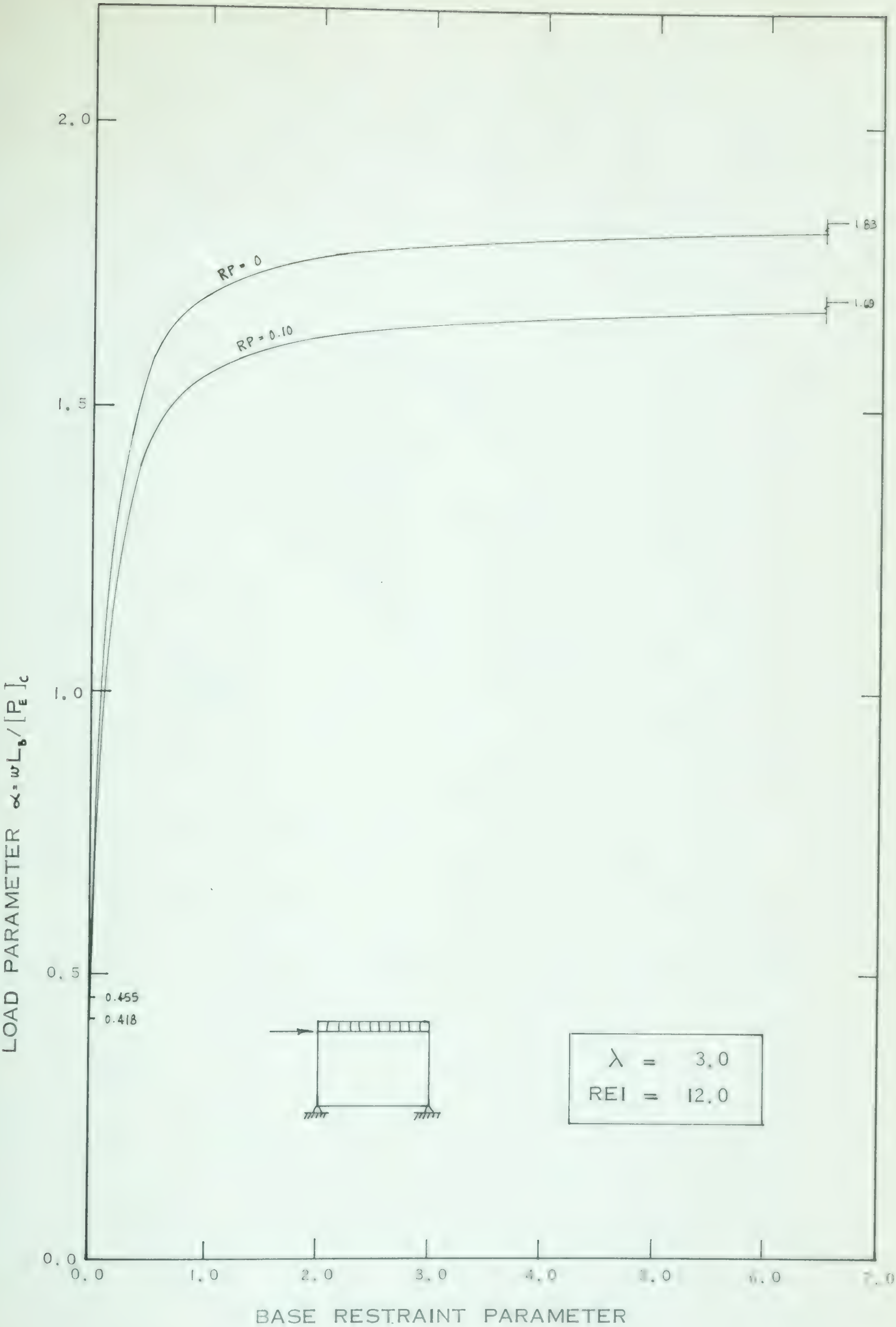


FIGURE 4 – 59 LOAD vs BASE RESTRAINT

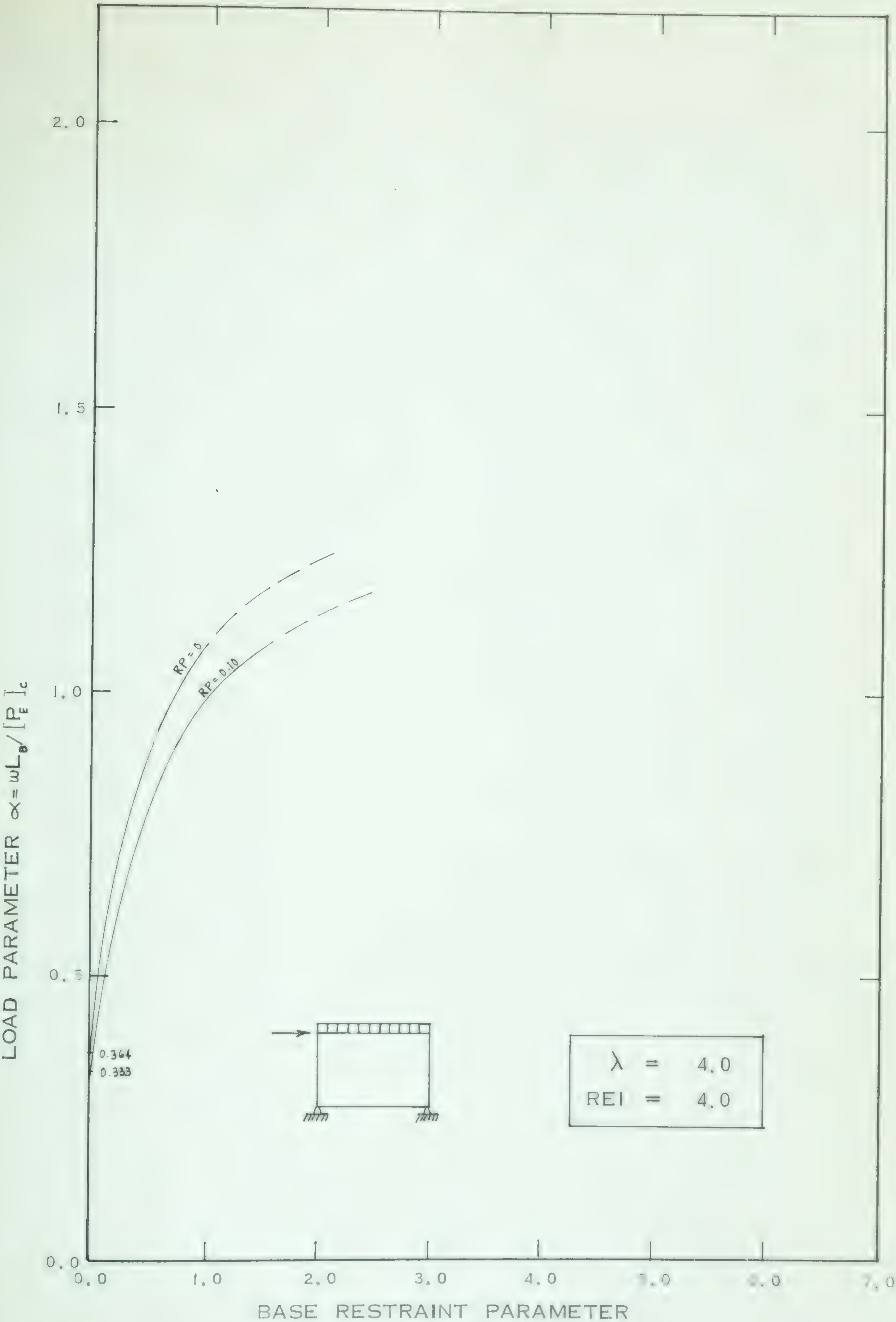


FIGURE 4 – 60 LOAD vs. BASE RESTRAINT

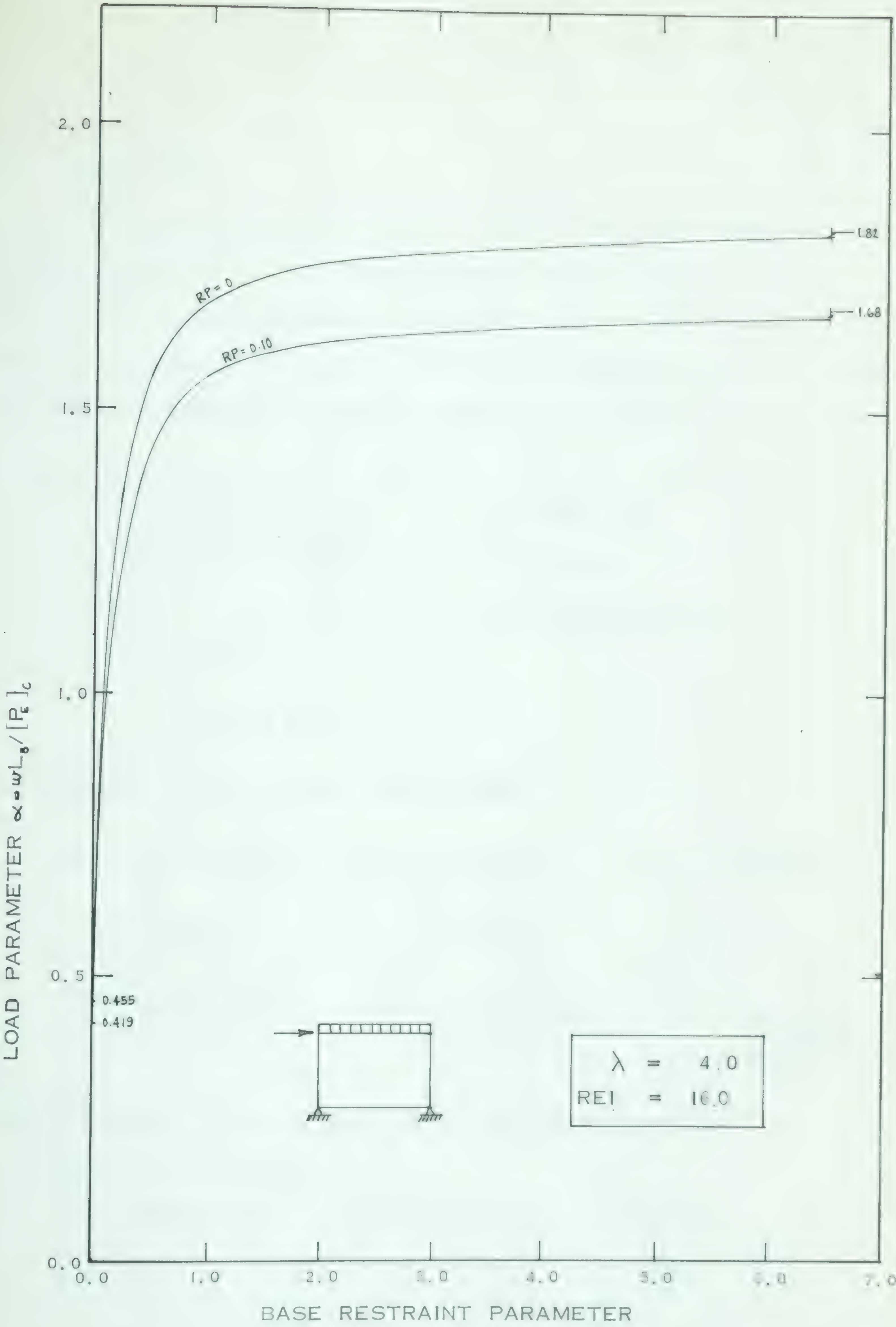
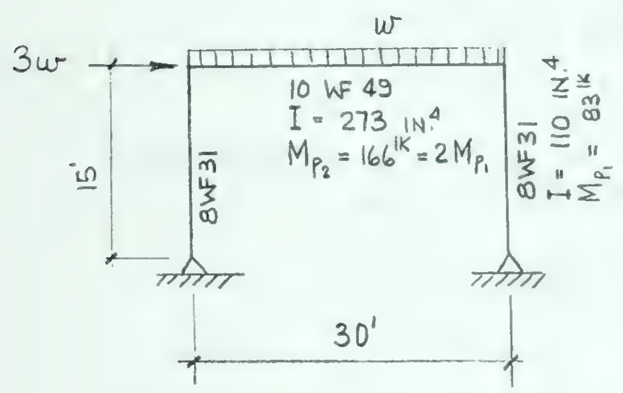


FIGURE 4 – 61 LOAD vs. BASE RESTRAINT

4-5 DESIGN EXAMPLE

To illustrate the use in actual design of the elastic critical loads derived in this thesis, the following design example is given. Using Merchant's empirical approach (as discussed on page 7), an estimate of the true failure load of this example frame will be obtained. Preliminary analysis indicates the horizontal load will be about 10% of the total vertical load.



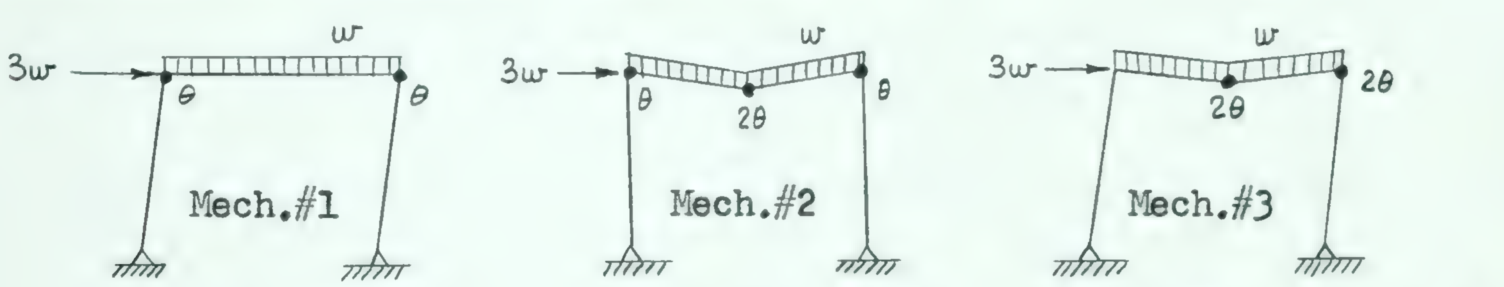
$E = 30 \times 10^3$ ksi

$\sigma_y = 33$ ksi

Neglect frame weight.

FIGURE 4-62

(A) Ultimate Strength - Simple Plastic Theory



$3\,w(15\theta) = 2M_{P_1}\,\theta$
$w = 0.0445\ M_{P_1}$

$30\,w(7.5\theta) = 6M_{P_1}\,\theta$
$w = 0.0266\ M_{P_1}$

$3\,w(15\theta) + 30\,w(7.5\theta) = 6M_{P_1}\,\theta$
$w = 0.0222\ M_{P_1}$

Mech. #3 governs. Hence, ultimate load $w_u = 0.0222(83) = 1.84^{K/'}$.

(B) Elastic Critical Load

$\lambda = 30/15 = 2.0 \qquad \text{REI} = 273/110 = 2.5 \qquad \text{RP} = 0.10$

Interpolating between figures 4-7 and 4-8, $\alpha = \dfrac{wL_B}{(P_E)_c} = 0.345$

$$w = 0.345\,(P_E)_c = \frac{0.345(\pi^2)(30\times10^3)(110)}{30(15)^2(144)} = 11.53^{K/'} = w_{CR}$$

(C) Actual Load-carrying Capacity

The best available estimate of the true failure load can be obtained by the application of Merchant's empirical equation (discussed on page 7).

$$\frac{1}{W_f} = \frac{1}{W_{CR}} + \frac{1}{W_U}$$

where W_f = true failure load
W_{CR} = elastic critical load
W_U = simple plastic collapse load

Therefore, $\frac{1}{W_f} = \frac{1}{11.53} + \frac{1}{1.84}$

$W_f = 1.59^{K/I}$ = true failure load.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The following general conclusions regarding the results of this investigation can be made:

(1) A slope-deflection analysis using an electronic digital computer is a suitable method for solving stability problems for frames subjected to axial loads and primary bending moments. However, the mathematical solution became unstable for the frame with fixed bases when the length of beam was large relative to the column length, and when the beam rigidity was small relative to the column rigidity. Similar difficulties may be encountered in future stability investigations of other building frames.

(2) A fixed-base frame of the type considered in this thesis will sustain a elastic critical load about four times greater than the corresponding hinged-base frame, for a given value of horizontal load. The load-deflection curves for these two frames are similar except for this ratio of loads. When horizontal load is present, both frames exhibit large sway deflections before the critical load is reached.

(3) Conclusions (4), (5) and (6) are applicable to both the hinged and fixed-base frames.

(4) The elastic critical load decreases with increasing horizontal load, the frame dimensions and ratio of member rigidities remaining constant. The percentage reduction from the case of no horizontal load is about the same for all combinations of variables investigated.

(5) The elastic critical load decreases at an increasing rate as the length of beam increases with respect to the length of columns, the horizontal load and ratio of member rigidities remaining constant.

(6) The elastic critical load increases at a decreasing rate as the beam rigidity increases with respect to the column rigidity, the horizontal load and frame dimensions remaining constant.

(7) The use of a so-called "large-deflection theory" results in higher load parameter values being obtained for a given sway rotation than those obtained by the small-deflection theory. This result is most pronounced for large sway rotations.

(8) The presence of even a small amount of base restraint increases the critical load considerably above that of a hinged base frame. This amount of base restraint is available under presently used column bases that are commonly assumed in design to be hinged bases.

The following fields of interest are suggested for future work in the elastic stability of frames subjected to axial loads and primary bending moments:

(1) The critical loads of portal frames under the action of uniform vertical and horizontal loadings may be investigated.

(2) The critical loads of multi-bay, multi-story rectangular building frames subjected to horizontal and vertical loads may be investigated.

(3) The critical loads of gable frames subjected to various combinations of loading may be investigated.

(4) The effect of gusset plates in the corners of these frames may also be investigated. Stability functions for members with gusset plates at the ends have been developed.(12)

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APPENDIX A

STIFFNESS AND CARRY-OVER FACTORS

Consider a member of length L and flexural rigidity EI. Define forces P and F, moments M_A and M_B, and end rotation θ_B positive as shown in figure A-1.

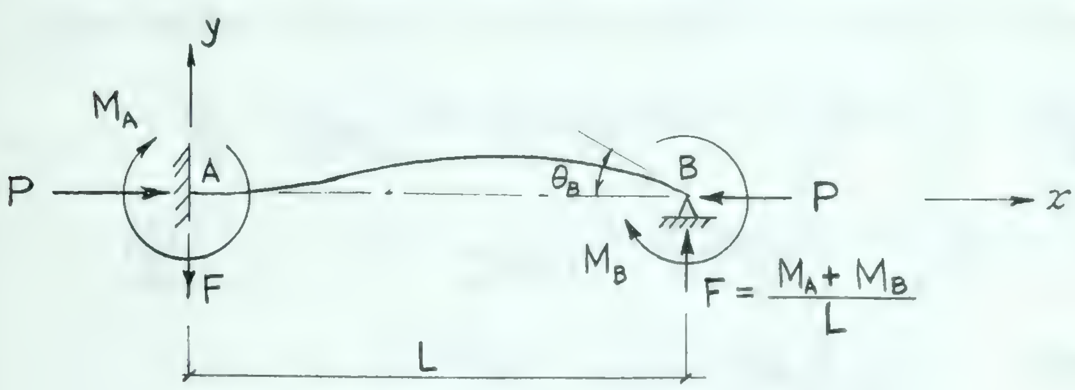


FIGURE A-1

By definition, the stiffness factor $K = SEI/L$ is the moment required at the near end of a member to rotate the near end through a unit angle. The carry-over factor C is the ratio of the moment induced at the far end to the moment applied at the near end. The stiffness coefficient S and carry-over factor C are then given by the relationships $M = SEI\theta_B/L$ and $M_A = CM_B$, and will be calculated for a member with one end fixed, as shown in Figure A-1.

The differential equation expressing the curvature of this member is based on the well-known relationship $EI \frac{d^2y}{dx^2} = M$ and hence becomes:

$$EI \frac{d^2y}{dx} = +M_A - \frac{(M_A + M_B)x}{L} - Py$$

The general solution to this equation is:

$$y = A \mathrm{Sin}\, kx + B \mathrm{Cos}\, kx + \frac{1}{P}\left[M_A - \frac{(M_A + M_B)x}{L} \right] \quad \text{where } k = \sqrt{\frac{P}{EI}} \text{ and}$$

A and B are constants to be determined by the boundary conditions.

Applying the boundary conditions of $y = 0$ at $x = 0$ and $x = L$, the general solution becomes:

$$y = -\frac{1}{k^2 EI}\left\{\left[\frac{\mathrm{Sin}\, k(L-x)}{\mathrm{Sin}\, kL} - \frac{(L-x)}{L}\right] M_A + \left[\frac{x}{L} - \frac{\mathrm{Sin}\, kx}{\mathrm{Sin}\, kL}\right] M_B\right\}$$

To obtain the carry-over factor C, it is necessary to obtain the value of M_A/M_B for the slope at A equal to zero, as follows:

$$\frac{dy}{dx} = -\frac{1}{Lk^2 EI}\left\{\left[\frac{-kL\, \mathrm{Cos}\, k(L-x)}{\mathrm{Sin}\, kL} + 1\right] M_A + \left[1 - \frac{kL\, \mathrm{Cos}\, kx}{\mathrm{Sin}\, kL}\right] M_B\right\}$$

$$\left(\frac{dy}{dx}\right)_{x = 0} = 0 = -\frac{1}{Lk^2 EI}\left\{\left[-\frac{kL\, \mathrm{Cos}\, kL}{\mathrm{Sin}\, kL} + 1\right] M_A + \left[1 - \frac{kL}{\mathrm{Sin}\, kL}\right] M_B\right\}$$

$$\therefore \boxed{C = \frac{M_A}{M_B} = \frac{\phi - \mathrm{Sin}\, \phi}{\mathrm{Sin}\, \phi - \phi\, \mathrm{Cos}\, \phi}} \qquad \text{where } \phi = kL$$

To obtain the stiffness factor K, it is necessary to find the slope at B and set this equal to unity while the slope at A remains zero, as follows:

When $\frac{dy}{dx} = 0$ at $x = 0$, the general formula for slope becomes

$$\frac{dy}{dx} = -\frac{1}{Lk^2 EI}\left\{\left[\frac{-kL\, \mathrm{Cos}\, k(L-x)}{\mathrm{Sin}\, kL} + 1\right]\left[\frac{kL - \mathrm{Sin}\, kL}{\mathrm{Sin}\, kL - kL\, \mathrm{Cos}\, kL}\right] M_B + \left[1 - \frac{kL\, \mathrm{Cos}\, kL}{\mathrm{Sin}\, kL}\right] M_B\right\}$$

$$\left(\frac{dy}{dx}\right)_{x = L} = \theta_B = \frac{-M_B}{Lk^2 EI}\left\{\left[\frac{-kL}{\mathrm{Sin}\, kL} + 1\right]\left[\frac{kL - \mathrm{Sin}\, kL}{\mathrm{Sin}\, kL - kL\, \mathrm{Cos}\, kL}\right] + \left[1 - \frac{kL\, \mathrm{Cos}\, kL}{\mathrm{Sin}\, kL}\right]\right\}$$

$$\theta_B = -\frac{M_B}{kEI}\left[\frac{2 - 2\, \mathrm{Cos}\, kL - kL\, \mathrm{Sin}\, kL}{\mathrm{Sin}\, kL - kL\, \mathrm{Cos}\, kL}\right]$$

Letting $\theta_B = -1$, $K = M_B = \phi\left[\frac{\mathrm{Sin}\, \phi - \phi\, \mathrm{Cos}\, \phi}{2 - 2\, \mathrm{Cos}\, \phi - \phi\, \mathrm{Sin}\, \phi}\right]\frac{EI}{L}$

$$\therefore \boxed{S = \phi\left[\frac{\mathrm{Sin}\, \phi - \phi\, \mathrm{Cos}\, \phi}{2 - 2\, \mathrm{Cos}\, \phi - \phi\, \mathrm{Sin}\, \phi}\right]}$$

The above C and S functions were derived for the case of axial compression. Similar derivations for the case of axial tension yield the following expressions for C and S:

$$\boxed{\begin{aligned} C &= \frac{\phi - \text{Sinh}\,\phi}{\text{Sinh}\,\phi - \phi\,\text{Cosh}\,\phi} \\ S &= \phi\left[\frac{\text{Sinh}\,\phi - \phi\,\text{Cosh}\,\phi}{2\,\text{Cosh}\,\phi - 2 - \phi\,\text{Sinh}\,\phi}\right] \end{aligned}}$$

If the axial load is zero, it would be expected that the C and S functions would reduce to the conventional values of 0.5 and 4.0 respectively. However, the substitution of $\phi = 0$ causes all of the above expressions to reduce to zero. This apparent discrepancy can be overcome by the application of L'Hospitals rule, which reads as follows:

$$\text{Lim}_{x \to 0} \frac{f(x)}{g(x)} = \text{Lim}_{x \to 0} \frac{f'(x)}{g'(x)}$$

Therefore, successive differentiation of the expressions for stiffness coefficient and carry-over factor with respect to ϕ and the substitution of $\phi = 0$ will result in C = 0.5 and S = 4.0, which are the correct values when the axial load is zero.

It must be noted that the stiffness and carry-over factors are based on the equation $M/EI = d^2y/dx^2$. This equation is valid only when the difference between the total length along a member and its corresponding chord length is negligible. Therefore, these stiffness and carry-over factors are not strictly correct for a true large deflection theory.

APPENDIX B

SLOPE-DEFLECTION EQUATIONS

The resultant end moment on a frame member under the applied loading is made up of the following components:

(A) Moments due to rotations of the ends of the frame member

(B) Moments due to lateral displacement of the ends of the member

(C) Moments due to effects of lateral loads applied to the frame member under consideration

The magnitude of the axial load present in the member affects the resultant end moment and must be considered in any stability analysis. It is a well-known fact that the effects of axial load and moment cannot be evaluated separately and then combined to obtain a correct answer. The true resultant end moment must be obtained by considering the effect of axial load on each of the component moments above, and then superposing these components by direct addition. This is the basic principle of the slope-deflection equations.

The component moments making up the resultant end moment will now be considered separately in the following sections. Note that the slope-deflection equations neglect the effects of shear or axial deformations, which are relatively small compared to flexural deformations.

(A) Moments due to end rotations

Consider a member of length L and flexural rigidty EI. Define

forces P and F, moments M_A and M_B, and end rotations θ_A and θ_B positive as shown in figure B-1.

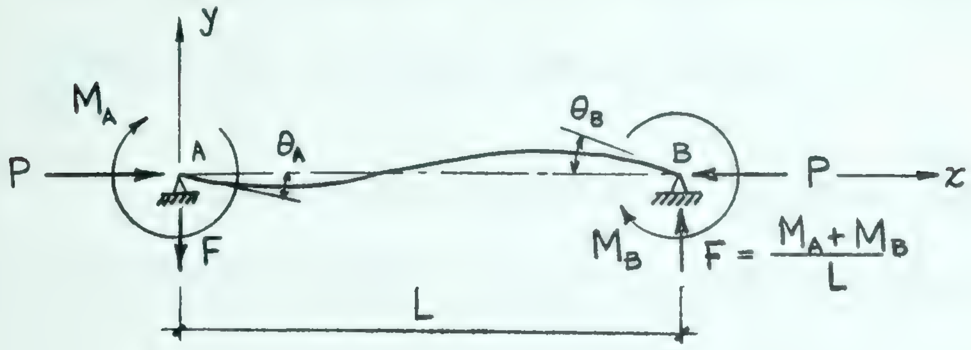


FIGURE B-1

As given in APPENDIX A, the general solution to the differential equation defining the equilibrium configuration of this member is $y = \frac{1}{k^2 EI}\left\{\left[\frac{\sin k(L-x)}{\sin kL} - \frac{(L-x)}{L}\right] M_A + \left[\frac{x}{L} - \frac{\sin kx}{\sin kL}\right] M_B\right\}$ where $k = \sqrt{\frac{P}{EI}}$

The general expression for slope becomes

$$\frac{dy}{dx} = \frac{1}{Lk^2 EI}\left\{\left[-\frac{kL \cos k(L-x)}{\sin kL} + 1\right] M_A + \left[1 - kL\frac{\cos kx}{\sin kL}\right] M_B\right\}$$

Then, $$\theta_A = \left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{Lk^2 EI}\left\{\left[-\frac{kL \cos kL}{\sin kL} + 1\right] M_A + \left[1 - \frac{kL}{\sin kL}\right] M_B\right\}$$

$$\theta_B = \left(\frac{dy}{dx}\right)_{x=L} = \frac{1}{Lk^2 EI}\left\{\left[-\frac{kL}{\sin kL} + 1\right] M_A + \left[1 - \frac{kL \cos kL}{\sin kL}\right] M_B\right\}$$

Cross-multiplying,

$$\theta_A Lk^2 EI \sin kL = (\sin kL - kL \cos kL)M_A + (\sin kL - kL)M_B \quad (1)$$

$$\theta_B Lk^2 EI \sin kL = (\sin kL - kL)M_A + (\sin kL - kL \cos kL)M_B \quad (2)$$

From (1),

$$M_A = \frac{\theta_A Lk^2 EI \sin kL - (\sin kL - kL)M_B}{\sin kL - kL \cos kL}$$

Substituting into (2) and manipulating terms,

$$M_B = kL\left[\frac{\sin kL - kL \cos kL}{2 - 2 \cos kL - kL \sin kL}\right]\frac{EI}{L}\theta_B$$

$$+ kL\left[\frac{\sin kL - kL \cos kL}{2 - 2 \cos kL - kL \sin kL}\right]\left[\frac{kL - \sin kL}{\sin kL - kL \cos kL}\right]\frac{EI}{L}\theta_A$$

or $$\boxed{M_B = K\theta_B + KC\theta_A = K(\theta_B + C\theta_A) = \frac{SEI}{L}(\theta_B + C\theta_A)}$$

This expression was derived for the case of axial compression. A similar derivation for the case of axial tension will yield the same expression if the corresponding carry-over and stiffness factors for this case are utilized.

It should be noted that the axial load P as defined is the load in the member lying on a line joining the ends of the member. For a frame member subjected to sway, it is often more convenient to calculate the axial load as the load on a line joining the ends of the member in the undeflected configuration. For small sway deflections, these axial loads are nearly equal, and so little error is introduced. However, for large deflections, the axial load lying along a line joining the ends of the member in the deflected configuration must be used. This concept is illustrated in the analysis of the portal frame in APPENDIX C.

(B) Moments due to lateral displacement

Consider the same member subjected to a sway deflection , as shown in figure B-2.

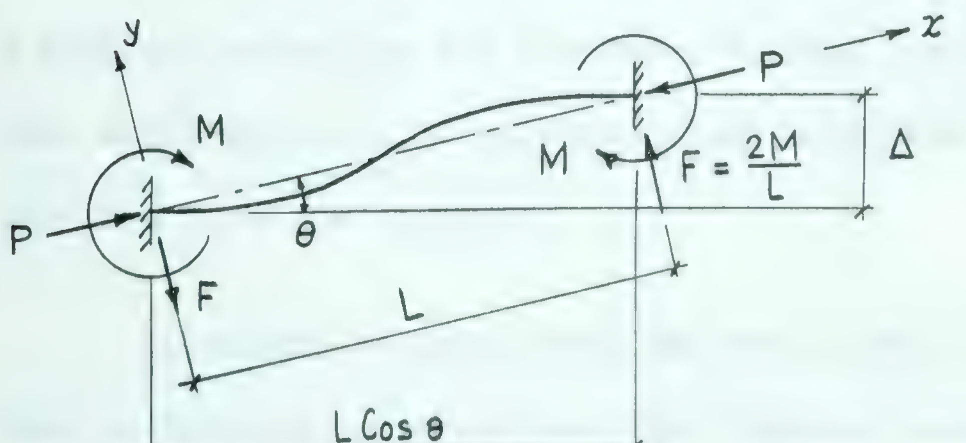


FIGURE B-2

The differential equation defining the equilibrium configuration of this member reads as follows:

$$EI \frac{d^2y}{dx^2} = M - Py - \frac{2Mx}{L}$$

The general solution to this equation is:

$y = A \operatorname{Sin} kx + B \operatorname{Cos} kx + \frac{1}{P}\left[M - \frac{2Mx}{L}\right]$ where $k = \sqrt{\frac{P}{EI}}$ and A and B are constants to be determined by the boundary conditions.

Applying the boundary conditions of $y = 0$ at $x = 0$ and $x = L$, the general solution becomes:

$$y = \frac{M}{k^2 EI}\left[\frac{\operatorname{Sin} kx}{\operatorname{Tan} kL} + \frac{\operatorname{Sin} kx}{\operatorname{Sin} kL} - \operatorname{Cos} kx - \frac{2x}{L} + 1\right]$$

The general expression for slope them becomes:

$$\frac{dy}{dx} = \frac{M}{k^2 EI}\left[\frac{k \operatorname{Cos} kx}{\operatorname{Tan} kL} + \frac{k \operatorname{Cos} kx}{\operatorname{Sin} kL} + k \operatorname{Sin} kx - \frac{2}{L}\right]$$

At $x = 0$ $\quad \frac{dy}{dx} = -\operatorname{Tan} \Theta = \frac{M}{k^2 EI}\left[\frac{k}{\operatorname{Tan} kL} + \frac{k}{\operatorname{Sin} kL} - \frac{2}{L}\right]$

Cross-multiplying and manipulating terms,

$$M = \emptyset\left[\frac{\operatorname{Sin} \emptyset - \emptyset \operatorname{Cos} \emptyset}{2 - 2 \operatorname{Cos} \emptyset - \emptyset \operatorname{Sin} \emptyset}\right]\left[\frac{EI}{L}\right]\left[1 + \frac{\emptyset - \operatorname{Sin} \emptyset}{\operatorname{Sin} \emptyset - \emptyset \operatorname{Cos} \emptyset}\right] \operatorname{Tan} \Theta$$

where $\emptyset = kL$ $\quad \therefore \boxed{M = K\,(1+C) \operatorname{Tan} \Theta = \frac{SEI}{L}\,(1+C) \operatorname{Tan} \Theta}$

This expression is derived for the case of axial compression. A similar derivation for the case of axial tension will yield the same expression if the corresponding carry-over and stiffness factors for this case are utilized.

It should be noted that the axial load P as defined is the load in the member lying on a line joining the ends of the member.

For small sway deflections, little error is introduced if the axial load used is that lying on a line joining the ends of the member in the undeflected configuration. Furthermore, for small sway deflections, Tan θ is effectively equal to θ, and the moment due to sway can be taken as $M = K(1+C)\theta$.

(C) Moments due to lateral load

This component moment is simply the fixed-end moment for the member, since end rotations are considered under part (A). The fixed end moment is a function of the magnitude of the axial load and the magnitude and distribution of the lateral load.

In general, for a prismatic member,

$$FEM_{AB} = -K(\theta_{AS} + C\theta_{BS})$$

where θ_{AS} and θ_{BS} are the rotations at the ends of the frame member when considered as a simply-supported bean-column subjected to axial and lateral loadings. This relation is an application of the results of part (A), since the ends of the member are rotated from their simply-supported position to that of no end rotation in order to get the end moments.

It is apparent that expressions for fixed end moment depend on the type of lateral load distribution. Since uniform lateral loads only are considered in this thesis, this case only will be considered here.

For a member subjected to uniform lateral load w and no axial load, as assumed in conventional frame analysis,

$$\Theta_{AS} = -\Theta_{BS} = \frac{\omega L^3}{24EI}$$

$$K = \frac{4EI}{L} \qquad C = 0.5$$

$$\therefore \; FEM_{AB} = -\frac{4EL}{L}\left[\frac{\omega L^3}{24EI}\right](1-0.5) = -\frac{\omega L^2}{12}$$

$$FEM_{BA} = +\frac{\omega L^2}{12}$$

Consider further a member subjected to uniform lateral load ω and axial compressive force P, as shown in figure B-3.

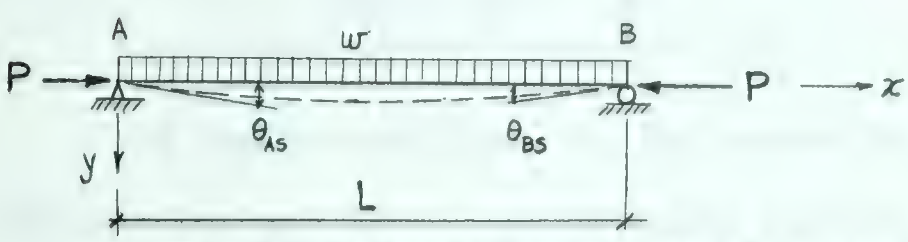


FIGURE B-3

The expressions for Θ_{AS} and Θ_{BS} can be derived by consideration of the differential equation (7)

$$EI\frac{d^4y}{dx^4} + P\frac{d^2y}{dx^2} = \omega$$

The general solution to this equation is:

$$Y = A \text{ Sin } kx + B \text{ Cos } kx + Cx + D + \frac{\omega x^2}{2P} \quad \text{where } k = \sqrt{\frac{P}{EI}}$$

and A, B, C, and D are constants to be determined by the boundary conditions.

$$\frac{dy}{dx} = Ak \text{ Cos } kx - Bk \text{ Sin } kx + C + \frac{\omega x}{P}$$

$$\frac{d^2y}{dx^2} = -Ak^2 \text{ Sin } kx - Bk^2 \text{ Cos } kx + \frac{\omega}{P}$$

Applying the boundary conditions of $y = 0$ and $d^2y/dx^2 = 0$ at both $x = 0$ and $x = L$, the general solution becomes

$$y = \frac{w}{k^2P}\left\{\left[\frac{1 - \cos kL}{\sin kL}\right](\sin kx) + \cos kx - 1\right\} - \frac{wx}{2P}(L-x)$$

$$\frac{dy}{dx} = \frac{w}{kP}\left\{\left[\frac{1 - \cos kL}{\sin kL}\right](\cos kx) - \sin kx\right\} - \frac{wL}{2P} + \frac{wx}{P}$$

Then, $$\left(\frac{dy}{dx}\right)_{x = 0} = \frac{w}{kP}\left[\frac{1 - \cos kL}{\sin kL}\right] - \frac{wL}{2P} = \theta_{AS}$$

or $$\theta_{AS} = \frac{wL}{2P}\left[\frac{2}{\phi} \tan \frac{\phi}{2} - 1\right] = -\theta_{BS} \quad \text{where } \phi = kL$$

Therefore, for a member subjected to a uniform lateral load w and an axial compressive force P,

$$FEM_{AB} = -\frac{SEI}{L}\left[\frac{WL}{2P}(1-C)\right]\left[\frac{2}{\phi} \tan \frac{\phi}{2} - 1\right]$$

or $$\boxed{FEM_{AB} = -\frac{S(1-C)}{2\phi^2}\left[\frac{2}{\phi} \tan \frac{\phi}{2} - 1\right] wL^2}$$

A similar analysis for the case of a member subjected to a uniform lateral load w and an axial tension force P will yield the following expression for fixed end moment, using the corresponding stiffness and carry-over factors for this case:

$$\boxed{FEM_{AB} = -\frac{S(1-C)}{2\phi^2}\left[1 - \frac{2}{\phi} \tanh \frac{\phi}{2}\right] wL^2}$$

These expressions are strictly correct only when the axial load used is that lying on a line joining the ends of the member in the deflected configuration, the lateral load used is that perpendicular to this stated axial force, and the rigidity EI is uniform throughout the entire length.

APPENDIX C

STRUCTURAL ANALYSES OF FRAME

Consider the portal frame of figure C-1 with the hinged column bases, and assume that the deformations in the equilibrium configuration of figure C-1 are small but finite.

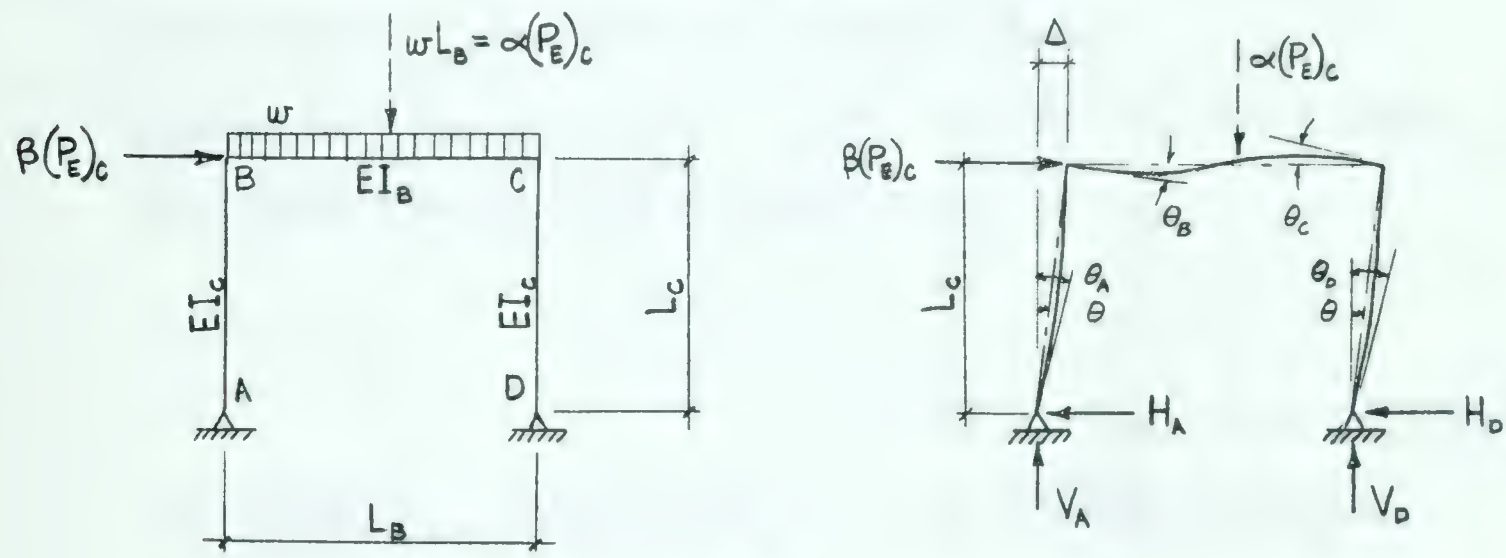


FIGURE C-1

The slope deflection equations are written as follows:

$$M_{AB} = K_{AB}(\theta_A + C_{AB}\theta_B) - K_{AB}(1 - C_{AB})\,\theta$$

$$M_{BA} = K_{AB}(\theta_B + C_{AB}\theta_A) - K_{AB}(1 + C_{AB})\,\theta$$

$$M_{BC} = K_{BC}(\theta_B + C_{BC}\theta_C) - FEM$$

$$M_{CB} = K_{BC}(\theta_C + C_{BC}\theta_B) + FEM$$

$$M_{CD} = K_{CD}(\theta_C + C_{CD}\theta_D) - K_{CD}(1 + C_{CD})\,\theta$$

$$M_{DC} = K_{CD}(\theta_D + C_{CD}\theta_C) - K_{CD}(1 + C_{CD})\,\theta$$

Summing moments about A,

$$K_{AB}(\theta_A + C_{AB}\theta_B) = K_{AB}(1 + C_{AB})\,\theta \quad \text{or} \quad \theta_A = (1 + C_{AB})\,\theta - C_{AB}\theta_B$$

Summing moments about B,

$$K_{AB}(\theta_B + C_{AB}\theta_A) - K_{AB}(1 + C_{AB})\theta + K_{BC}(\theta_B + C_{BC}\theta_C) - FEM = 0$$

Substituting θ_A from above and collecting like terms,

$$\left[K_{AB}(1 - C_{AB}^2) + K_{BC}\right]\theta_B + \left[K_{BC}C_{BC}\right]\theta_C - \left[K_{AB}(1 - C_{AB}^2)\right]\theta = FEM \qquad (1)$$

Summing moments about D,

$$K_{CD}(\theta_D + C_{CD}\theta_C) = K_{CD}(1+C_{CD})\theta \quad \text{or} \quad \theta_D = (1+C_{CD})\theta - C_{CD}\theta_C$$

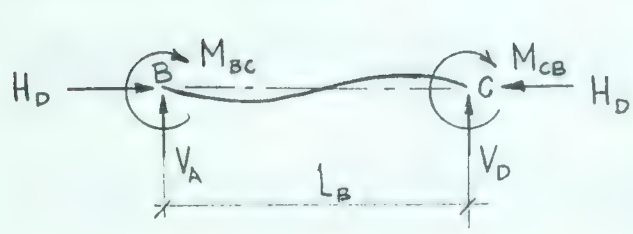
Summing moments about C,

$$K_{BC}(\theta_C + C_{BC}\theta_B) + FEM + K_{CD}(\theta_C + C_{CD}\theta_D) - K_{CD}(1 + C_{CD})\theta = 0$$

Substituting θ_D from above and collecting terms,

$$\left[K_{BC}C_{BC}\right]\theta_B + \left[K_{CD}(1 - C_{CD}^2) + K_{BC}\right]\theta_C - \left[K_{CD}(1 - C_{CD}^2)\right]\theta = -FEM \qquad (2)$$

The equilibrium equations for beam BC are:



$$+\circlearrowright \ \Sigma M_B = 0 = M_{BC} + M_{CB} + \alpha(P_E)_c \frac{L_B}{2} - V_D L_B$$

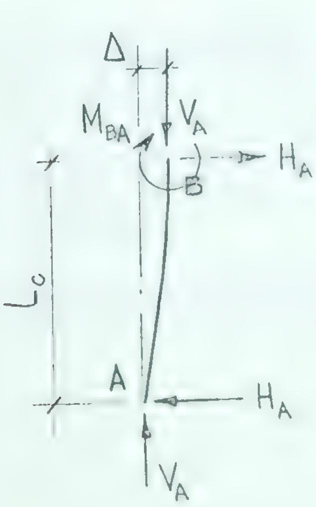
$$V_D = \frac{\alpha(P_E)_c}{2} + \frac{(M_{BC} + M_{CB})}{L_B}$$

$$+\downarrow \ \Sigma V = 0 = \alpha(P_E)_c - V_A - V_D$$

$$V_A = \frac{\alpha(P_E)_c}{2} - \frac{(M_{BC} + M_{CB})}{L_B}$$

FIGURE C-2

The equilibrium of column AB is as follows:



$$+\circlearrowright \ \Sigma M_B = 0 = M_{BA} + V_A\Delta + H_A L_c$$

$$H_A = \frac{-V_A\Delta - M_{BA}}{L_c}$$

FIGURE C-3

The equilibrium of column CD is similar and results in:

$$H_D = - \frac{V_D \Delta - M_{CD}}{L_C}$$

Summing the horizontal forces for the whole structure,

$$H_A + H_D = \beta(P_E)_C$$

$$V_A \Delta + M_{BA} + V_D \Delta + M_{CD} = - \beta(P_E)_C L_C$$

Substituting for V_A and V_D,

$$\alpha(P_E)_C \Delta + M_{BA} + M_{CD} = - \beta(P_E)_C L_C$$

Substituting for M_{BA} and M_{CD}, and subsequently for θ_A and θ_D, and collecting terms, the third governing equation is found to be:

$$\alpha(P_E)_C \Delta + \left[K_{AB}(1 - C_{AB}^2)\right]\theta_B + \left[K_{CD}(1 - C_{CD}^2)\right]\theta_C - \left[K_{AB}(1 - C_{AB}^2) + K_{CD}(1 - C_{CD}^2)\right]\theta = -\beta(P_E)_C L_C \quad ③$$

Rewrite equations ①, ② and ③ substituting:

$$K = SEI/L \quad \Delta = L_C\theta \quad \beta = RP\alpha \quad P_E = \pi^2 EI/L^2 \quad \lambda = L_B/L_C$$

and $\text{FEM} = \frac{S(1-C)F}{2}\omega L^2 = \frac{S(1-C)F}{2}\alpha(P_E)_C L_B = \frac{S(1-C)F}{2}\alpha\pi^2\lambda\left(\frac{EI}{L}\right)_C$

$$\left[S_{AB}\left(\frac{EI}{L}\right)_C(1-C_{AB}^2) + S_{BC}\left(\frac{EI}{L}\right)_B\right]\theta_B + \left[S_{BC}\left(\frac{EI}{L}\right)_B C_{BC}\right]\theta_C - \left[S_{AB}\left(\frac{EI}{L}\right)_C(1-C_{AB}^2)\right]\theta = \frac{S_{BC}(1-C_{BC})F_{BC}\alpha\pi^2\lambda}{2}\left(\frac{EI}{L}\right)_C \quad ①$$

$$\left[S_{BC}\left(\frac{EI}{L}\right)_B C_{BC}\right]\theta_B + \left[S_{CD}\left(\frac{EI}{L}\right)_C(1-C_{CD}^2) + S_{BC}\left(\frac{EI}{L}\right)_B\right]\theta_C - \left[S_{CD}\left(\frac{EI}{L}\right)_C(1 - C_{CD}^2)\right]\theta = \frac{-S_{BC}(1-C_{BC})F_{BC}\alpha\pi^2\lambda}{2}\left(\frac{EI}{L}\right)_C \quad ②$$

$$\alpha\pi^2\left(\frac{EI}{L}\right)_C\frac{\Delta}{L_C} + \left[S_{AB}\left(\frac{EI}{L}\right)_C(1-C_{AB}^2)\right]\theta_B + \left[S_{CD}\left(\frac{EI}{L}\right)_C(1-C_{CD}^2)\right]\theta_C - \left[S_{AB}\left(\frac{EI}{L}\right)_C(1-C_{AB}^2) + S_{CD}\left(\frac{EI}{L}\right)_C(1-C_{CD}^2)\right]\theta = -RP\alpha\pi^2\left(\frac{EI}{L}\right)_C\frac{L_C}{L_C} \quad ③$$

Divide equations ①, ②, ③, by $\left(\frac{EI}{L}\right)_B$ and let $\gamma = \frac{(EI/L)_C}{(EI/L)_B}$; then,

transpose terms to get the required form of the governing equations:

$$\left[S_{AB}\gamma(1-C_{AB}^2) + S_{BC}\right]\theta_B + \left[S_{BC}C_{BC}\right]\theta_C - \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2F_{BC}}{2}\right]\alpha = \left[S_{AB}\gamma(1-C_{AB}^2)\right]\theta \qquad (1)$$

$$\left[S_{BC}C_{BC}\right]\theta_B + \left[S_{CD}\gamma(1-C_{CD}^2) + S_{BC}\right]\theta_C + \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2F_{BC}}{2}\right]\alpha = \left[S_{CD}\gamma(1-C_{CD}^2)\right]\theta \qquad (2)$$

$$\left[S_{AB}(1-C_{AB}^2)\right]\theta_B + \left[S_{CD}(1-C_{CD}^2)\right]\theta_C + \left[\pi^2(RP+\theta)\right]\alpha = \left[S_{AB}(1-C_{AB}^2) + S_{CD}(1-C_{CD}^2)\right]\theta \qquad (3)$$

For the calculation of the S, C, and F functions, the axial forces in the members must be known.

The axial force in column AB is effectively:

$P_{AB} = V_A = \dfrac{\alpha(P_E)_C}{2} - \dfrac{(M_{BC}+M_{CB})}{L_B}$ which after substitution becomes:

$$P_{AB} = \frac{\alpha(P_E)_C}{2} - S_{BC}\left(\frac{EI}{L}\right)_B \frac{(1+C_{BC})(\theta_B+\theta_C)}{L_B}$$

Then $\rho_{AB} = \dfrac{P_{AB}}{(P_E)_C} = \dfrac{\alpha}{2} - \dfrac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}$

The axial force in column CD is:

$$P_{CD} = V_D = \frac{\alpha(P_E)_C}{2} + \frac{(M_{BC}+M_{CB})}{L_B}$$

Then $\rho_{CD} = \dfrac{\alpha}{2} + \dfrac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}$

The axial force in beam BC is:

$P_{BC} = H_D = \dfrac{-V_D\Delta - M_{CD}}{L_C}$ and by making the appropriate substitutions, $\rho_{BC} = \dfrac{P_{BC}}{(P_E)_B} = -\left[\dfrac{\alpha\gamma\lambda}{2} + \dfrac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2}\right]\theta - \dfrac{S_{CD}\gamma\lambda(1-C_{CD}^2)(\theta_C-\theta)}{\pi^2}$

The computer program as described in APPENDIX D solves the governing equations to obtain a value of α for given values of θ. An iterative solution must be used, since the governing equations contain the S, C, and F functions which depend on the axial load.

The above derivations have assumed small deformations of the frame. A so-called large deflection theory can also be developed by reference to figure C-4 which shows the frame in its equilibrium configuration under large deflections.

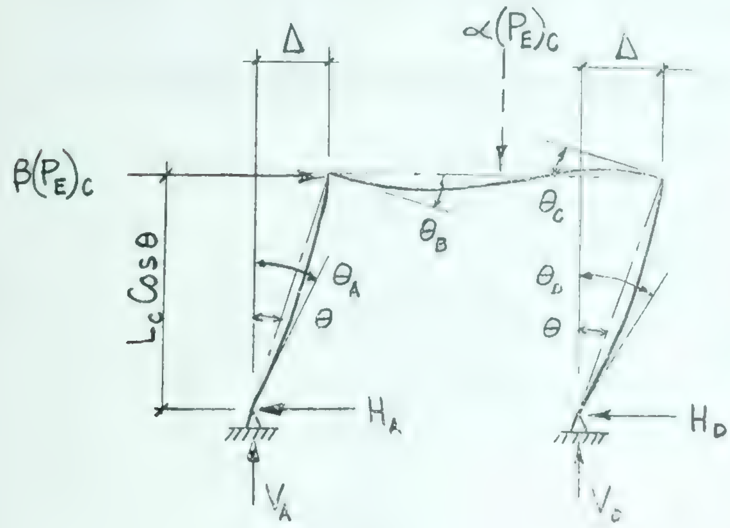


FIGURE C-4

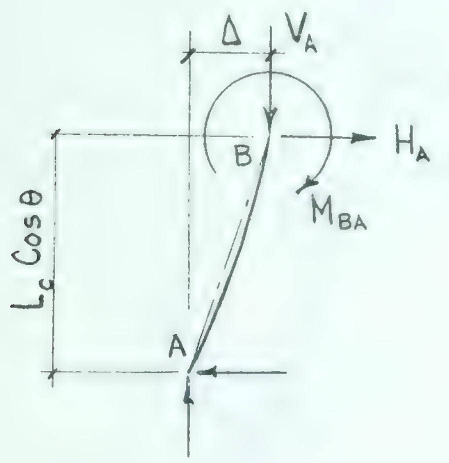
The difference between the large deflection and the small deflection theory considered in this thesis is basically two-fold:

(i) When writing the equilibrium equations for members with substantial sway deflections (such as column AB or CD), a reduced length LCosθ must be used when summing moments. For small deflections, LCosθ becomes effectively equal L, and the equilibrium equations become that of the small deflection theory.

(ii) When calculating the axial load for members with substantial sway deflections, the load on a line joining the ends of the member in the deflected configuration must be used. For small deflections, the load lying on a line joining the ends of the member in the undeflected configuration can be used with very little error.

Following this reasoning, the slope-deflection equations for this frame are the same as for the small deflection theory except θ

is replaced by Tanθ, as discussed under APPENDIX B. Hence, summing moments about points A,B,C, and D results in the same equations as the small deflection theory except θ is replaced by Tan θ. The equilibrium equations for beam BC remain exactly the same. The equilibrium of column AB changes, however, and is as follows:



$$\circlearrowright^{+} \quad \Sigma M_B = 0 = M_{BA} + V_A \Delta + H_A L_c \text{Cos}\theta$$

$$H_A = \frac{-V_A \Delta - M_{BA}}{L_c \text{ Cos } \theta}$$

FIGURE C-5

The equilibrium of column CD is similar and results in:

$$H_D = \frac{-V_D \Delta - M_{CD}}{L_c \text{ Cos } \theta}$$

Summing the horizontal forces for the whole structure follows the same routine as before, and results in the third governing equation which is:

$$\alpha (P_E)_c \Delta + \left[K_{AB}(1-C_{AB}^2)\right]\theta_B + \left[K_{CD}(1-C_{CD}^2)\right]\theta_c - \left[K_{AB}(1-C_{AB}^2) + K_{CD}(1-C_{CD}^2)\right] \text{Tan } \theta$$
$$= -\beta (P_E)_c \; L_c \text{ Cos } \theta \qquad (3)$$

Rewriting equations (1), (2), and (3) by making the same substitutions as before except $\Delta = L_c \text{Sin}\theta$, dividing by $(EI/L)_B$, and then transposing terms, the required form of the governing equations becomes:

$$\left[S_{AB}\gamma(1-C_{AB}^2) + S_{BC}\right]\theta_B + \left[S_{BC}\, C_{BC}\right]\theta_c - \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2}{2}F_{BC}\right]\alpha = \left[S_{AB}\gamma(1-C_{AB}^2)\right]\text{Tan }\theta \qquad (1)$$

$$\left[S_{BC}\, C_{BC}\right]\theta_B + \left[S_{CD}\gamma(1-C_{CD}^2) + S_{BC}\right]\theta_c + \left. S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2}{2}F_{BC}\right]\alpha = \left[S_{CD}\gamma(1-C_{CD}^2)\right]\text{Tan }\theta \qquad (2)$$

$$\left[S_{AB}(1-C_{AB}^2)\right]\theta_B + \left[S_{CD}(1-C_{CD}^2)\right]\theta_C + \left[\pi^2(RP\cos\theta + \sin\theta)\right]\alpha$$
$$= \left[S_{AB}(1-C_{AB}^2) + S_{CD}(1-C_{CD}^2)\right]\tan\theta \qquad (3)$$

The axial force in column AB is derived as follows, considering figure C-6.

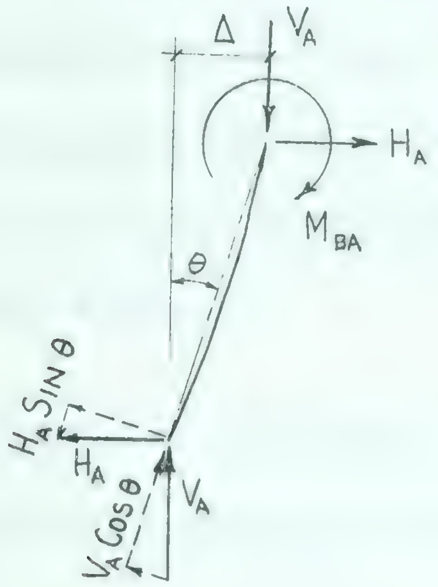


FIGURE C-6

$$P_{AB} = V_A\cos\theta - H_A\sin\theta$$
$$= \left[\frac{\alpha(P_E)_c}{2} - \frac{(M_{BC}+M_{CB})}{L_B}\right]\cos\theta + \left[\frac{\alpha(P_E)_c\,\Delta}{2L_c\cos\theta} - \frac{(M_{BC}+M_{CB})}{L_B L_c\cos\theta} + \frac{M_{BA}}{L_c\cos\theta}\right]\sin\theta$$

After substituting for M_{BC}, M_{CB} and M_{BA}, and simplifying, the expression becomes:

$$P_{AB} = \left[\frac{\alpha(P_E)_c}{2} - S_{BC}\left(\frac{EI}{L}\right)_B\frac{(1+C_{BC})(\theta_B+\theta_c)}{L_B}\right]\frac{1}{\cos\theta} + \left[S_{AB}\left(\frac{EI}{L}\right)_c\frac{(1-C_{AB}^2)(\theta_B-\tan\theta)}{L_c}\right]\tan\theta$$

Then, $\rho_{AB} = \dfrac{P_{AB}}{(P_E)_c}$

$$\rho_{AB} = \left[\frac{\alpha}{2} - \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_c)}{\pi^2\gamma\lambda}\right]\frac{1}{\cos\theta} + \left[\frac{S_{AB}(1-C_{AB}^2)(\theta_B-\tan\theta)}{\pi^2}\right]\tan\theta$$

A similar analysis for column CD yields:

$$\rho_{CD} = \left[\frac{\alpha}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_c)}{\pi^2\gamma\lambda}\right]\frac{1}{\cos\theta} + \left[\frac{S_{CD}(1-C_{CD}^2)(\theta_c-\tan\theta)}{\pi^2}\right]\tan\theta$$

For beam BC, $P_{BC} = H_D$ and

$$\rho_{BC} = -\left[\frac{\alpha\gamma\lambda}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2}\right]\text{Tan}\theta - \left[\frac{S_{CD}\gamma\lambda(1-C_{CD}^2)(\theta_C-\text{Tan}\theta)}{\pi^2}\right]\frac{1}{\text{Cos}\,\theta}$$

The small and large deflection theories have now been developed for the frame with the hinged bases. The development of the governing equations for the frames with fixed base and partial base fixity have been completed and are available at the Department of Civil Engineering of the University of Alberta. The analyses parallel those of the previous sections.

A summary of the governing equations for the frames considered in this thesis is outlined on the following pages.

Frame with Hinged Bases

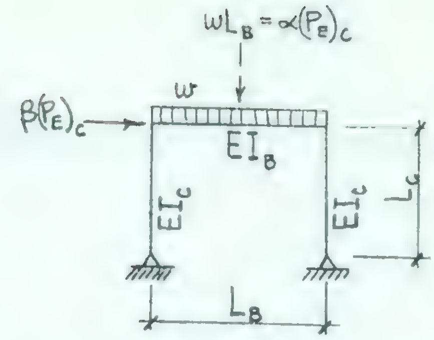


FIGURE C-7

(A) Small Deflection Theory

$$\left[S_{AB}\gamma(1-C_{AB}^2) + S_{BC}\right]\theta_B + \left[S_{BC}C_{BC}\right]\theta_C - \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2}{2}F_{BC}\right]\alpha = \left[S_{AB}\gamma(1-C_{AB}^2)\right]\theta$$

$$\left[S_{BC}C_{BC}\right]\theta_B + \left[S_{CD}\gamma(1-C_{CD}^2) + S_{BC}\right]\theta_C + \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2}{2}F_{BC}\right]\alpha = \left[S_{CD}\gamma(1-C_{CD}^2)\right]\theta$$

$$\left[S_{AB}(1-C_{AB}^2)\right]\theta_B + \left[S_{CD}(1-C_{CD}^2)\right]\theta_C + \left[\pi^2(RP+\theta)\right]\alpha = \left[S_{AB}(1-C_{AB}^2) + S_{CD}(1-C_{CD}^2)\right]\theta$$

$$\rho_{AB} = \frac{\alpha}{2} - \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}$$

$$\rho_{CD} = \frac{\alpha}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}$$

$$\rho_{BC} = -\left[\frac{\alpha\gamma\lambda}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2}\right]\theta - \frac{S_{CD}\gamma\lambda(1-C_{CD}^2)(\theta_C-\theta)}{\pi^2}$$

(B) Large Deflection Theory

$$\left[S_{AB}\gamma(1-C_{AB}^2) + S_{BC}\right]\theta_B + \left[S_{BC}C_{BC}\right]\theta_C - \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2}{2}F_{BC}\right]\alpha = \left[S_{AB}\gamma(1-C_{AB}^2)\right]\text{Tan }\theta$$

$$\left[S_{BC}C_{BC}\right]\theta_B + \left[S_{CD}\gamma(1-C_{CD}^2) + S_{BC}\right]\theta_C + \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2}{2}F_{BC}\right]\alpha = \left[S_{CD}\gamma(1-C_{CD}^2)\right]\text{Tan }\theta$$

$$\left[S_{AB}(1-C_{AB}^2)\right]\theta_B + \left[S_{CD}(1-C_{CD}^2)\right]\theta_C + \left[\pi^2(RP\text{ Cos }\theta + \text{Sin }\theta)\right]\alpha = \left[S_{AB}(1-C_{AB}^2) + S_{CD}(1-C_{CD}^2)\right]\text{Tan }\theta$$

$$\rho_{AB} = \left[\frac{\alpha}{2} - \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}\right]\frac{1}{\text{Cos }\theta} + \left[\frac{S_{AB}(1-C_{AB}^2)(\theta_B - \text{Tan }\theta)}{\pi^2}\right]\text{Tan }\theta$$

$$\rho_{CD} = \left[\frac{\alpha}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}\right]\frac{1}{\text{Cos }\theta} + \left[\frac{S_{CD}(1-C_{CD}^2)(\theta_C - \text{Tan }\theta)}{\pi^2}\right]\text{Tan }\theta$$

$$\rho_{BC} = -\left[\frac{\alpha\gamma\lambda}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2}\right]\text{Tan }\theta - \left[\frac{S_{CD}\gamma\lambda(1-C_{CD}^2)(\theta_C - \text{Tan }\theta)}{\pi^2}\right]\frac{1}{\text{Cos }\theta}$$

Frame with Fixed Bases

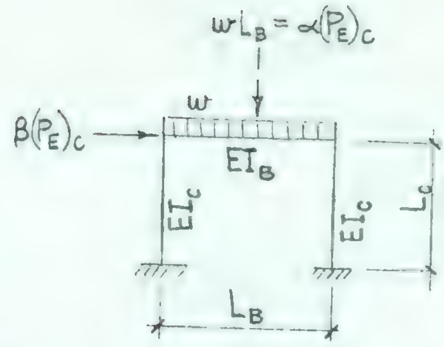


FIGURE C-8

(A) Small Deflection Theory

$$\left[S_{AB}\gamma + S_{BC}\right]\theta_B + \left[S_{BC}C_{BC}\right]\theta_C - \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2 F_{BC}}{2}\right]\alpha = \left[S_{AB}\gamma(1+C_{AB})\right]\theta$$

$$\left[S_{BC}C_{BC}\right]\theta_B + \left[S_{BC} + S_{CD}\gamma\right]\theta_C + \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2 F_{BC}}{2}\right]\alpha = \left[S_{CD}\gamma(1+C_{CD})\right]\theta$$

$$\left[S_{AB}(1+C_{AB})\right]\theta_B + \left[S_{CD}(1+C_{CD})\right]\theta_C + \left[\pi^2(RP+\theta)\right]\alpha = \left[2S_{AB}(1+C_{AB}) + 2S_{CD}(1+C_{CD})\right]\theta$$

$$\rho_{AB} = \frac{\alpha}{2} - \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\lambda\gamma}$$

$$\rho_{CD} = \frac{\alpha}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\lambda\gamma}$$

$$\rho_{BC} = -\left[\frac{\alpha\gamma\lambda}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2}\right]\theta - \frac{S_{CD}\lambda\gamma(1+C_{CD})(\theta_C-2\theta)}{\pi^2}$$

(B) Large Deflection Theory

$$\left[S_{AB}\gamma + S_{BC}\right]\theta_B + \left[S_{BC}C_{BC}\right]\theta_C - \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2 F_{BC}}{2}\right]\alpha = \left[S_{AB}\gamma(1+C_{AB})\right]\text{Tan }\theta$$

$$\left[S_{BC}C_{BC}\right]\theta_B + \left[S_{BC} + S_{CD}\gamma\right]\theta_C + \left[S_{BC}(1-C_{BC})\frac{\gamma\lambda\pi^2 F_{BC}}{2}\right]\alpha = \left[S_{CD}\gamma(1+C_{CD})\right]\text{Tan }\theta$$

$$\left[S_{AB}(1+C_{AB})\right]\theta_B + \left[S_{CD}(1+C_{CD})\right]\theta_C + \left[\pi^2(RP\text{ Cos }\theta + \text{Sin }\theta)\right]\alpha = \left[2S_{AB}(1+C_{AB}) + 2S_{CD}(1+C_{CD})\right]\text{Tan }\theta$$

$$\rho_{AB} = \left[\frac{\alpha}{2} - \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}\right]\frac{1}{\text{Cos }\theta} + \left[\frac{S_{AB}(1+C_{AB})(\theta_B - 2\text{Tan }\theta)}{\pi^2}\right]\text{Tan }\theta$$

$$\rho_{CD} = \left[\frac{\alpha}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}\right]\frac{1}{\text{Cos }\theta} + \left[\frac{S_{CD}(1+C_{CD})(\theta_C - 2\text{Tan }\theta)}{\pi^2}\right]\text{Tan }\theta$$

$$\rho_{BC} = -\left[\frac{\alpha\gamma\lambda}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2}\right]\text{Tan }\theta - \left[\frac{S_{CD}\gamma\lambda(1+C_{CD})(\theta_C - 2\text{Tan }\theta)}{\pi^2}\right]\frac{1}{\text{Cos }\theta}$$

<u>Frame with Partial Base Fixity</u>

Small Deflection Theory Only

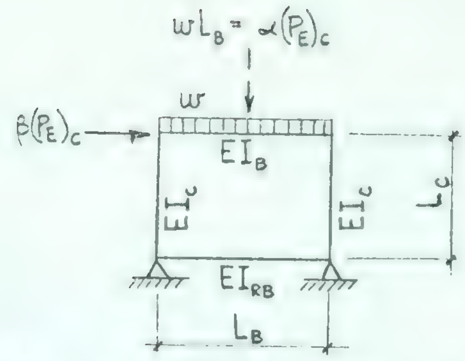


FIGURE C-9

$$\left[S_{AB}\gamma C_{AB} - \frac{(S_{AB}\gamma+S_{BC})(S_{AB}\gamma+4\eta)}{S_{AB}\gamma C_{AB}} - \frac{2\eta S_{BC}C_{BC}}{S_{CD}\gamma C_{CD}}\right]\theta_A - \left[\frac{2\eta(S_{AB}\gamma+S_{BC})}{S_{AB}\gamma C_{AB}} + \frac{S_{BC}C_{BC}(S_{CD}\ \ +4\eta)}{S_{CD}\gamma C_{CD}}\right]\theta_D$$

$$-\left[S_{BC}\ (1-C_{BC})\frac{\gamma\lambda\pi^2}{2}F_{BC}\right]\alpha = \left[S_{AB}\gamma(1+C_{AB}) - \frac{(S_{AB}\gamma+S_{BC})(1+C_{AB})}{C_{AB}} - S_{BC}C_{BC}\frac{(1+C_{CD})}{C_{CD}}\right]\theta$$

$$\left[\frac{-S_{BC}C_{BC}(S_{AB}\gamma+4\eta)}{S_{AB}\gamma C_{AB}} - \frac{2\eta(S_{BC}+S_{CD}\gamma)}{S_{CD}\gamma C_{CD}}\right]\theta_A + \left[S_{CD}\gamma C_{CD} - \frac{(S_{BC}+S_{CD}\gamma)(S_{CD}\gamma+4\eta)}{S_{CD}\gamma C_{CD}} - \frac{2\eta S_{BC}C_{BC}}{S_{AB}\gamma C_{AB}}\right]\theta_D$$

$$+\left[S_{BC}\ (1-C_{BC})\frac{\gamma\lambda\pi^2}{2}F_{BC}\right]\alpha = \left[S_{CD}\gamma(1+C_{CD}) - \frac{(S_{BC}+S_{CD}\gamma)(1+C_{CD})}{C_{CD}} - S_{BC}C_{BC}\frac{(1+C_{AB})}{C_{AB}}\right]\theta$$

$$\left[S_{AB}(1+C_{AB}) - \frac{(1+C_{AB})(S_{AB}\gamma+4\eta)}{\gamma C_{AB}} - \frac{2\eta(1+C_{CD})}{\gamma C_{CD}}\right]\theta_A + \left[S_{CD}(1+C_{CD}) - \frac{(1+C_{CD})(S_{CD}\gamma+4\eta)}{\gamma C_{CD}}\right.$$

$$\left.- \frac{2\eta(1+C_{AB})}{\gamma C_{AB}}\right]\theta_D + \left[\pi^2(RP+\theta)\right]\alpha = \left[2S_{AB}(1+C_{AB}) + 2S_{CD}(1+C_{CD}) - \frac{S_{AB}(1+C_{AB})^2}{C_{AB}}\right.$$

$$\left.- \frac{S_{CD}(1+C_{CD})^2}{C_{CD}}\right]\theta$$

$$\rho_{AB} = \frac{\alpha}{2} - \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}$$

$$\rho_{CD} = \frac{\alpha}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2\gamma\lambda}$$

$$\rho_{BC} = -\left[\frac{\alpha\gamma\lambda}{2} + \frac{S_{BC}(1+C_{BC})(\theta_B+\theta_C)}{\pi^2}\right]\theta - \frac{S_{CD}\gamma\lambda(1+C_{CD})(\theta_C+\theta_D-2\theta)}{\pi^2}$$

It should be noted that the results for all frames are reported in CHAPTER IV using the variable REI $= (EI)_B/(EI)_c$. Knowing γ and λ, REI is obtained from the relation REI $= \lambda/\gamma$.

APPENDIX D

COMPUTER PROGRAMS

The use of the University of Alberta's I.B.M. 1620 Electronic Digital Computer greatly facilitated solution of the governing equations given in APPENDIX C. Since the S, C, and F functions depend on the axial loads in the members which are not known until the governing equations are solved, an iterative solution is necessary to obtain the desired degree of accuracy. Values of S, C, and F of 4.0, 0.5, and $0.08\overline{3}$ respectively for no axial loads present were used to obtain the initial solution of the governing equations.

The flow diagram given in figure D-2 briefly describes the general sequence of operations that was introduced to the computer for the solution of the critical loads of the portal frames analyzed in this thesis. Basically, the computer was instructed to calculate successive values of the load parameter α for successive increments of the deflection parameter θ. The point on the load-deflection plot at which deflection increases with no increase in load defines the critical load, as in figure D-1.

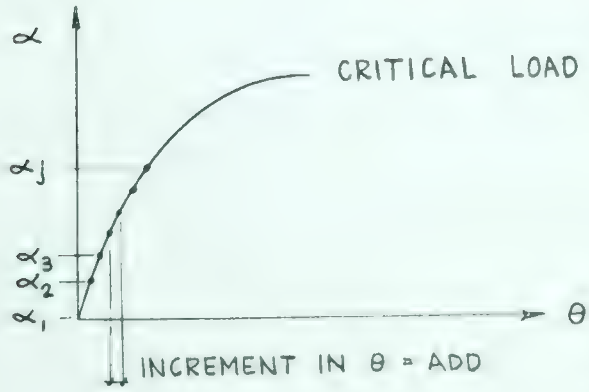


FIGURE D-1

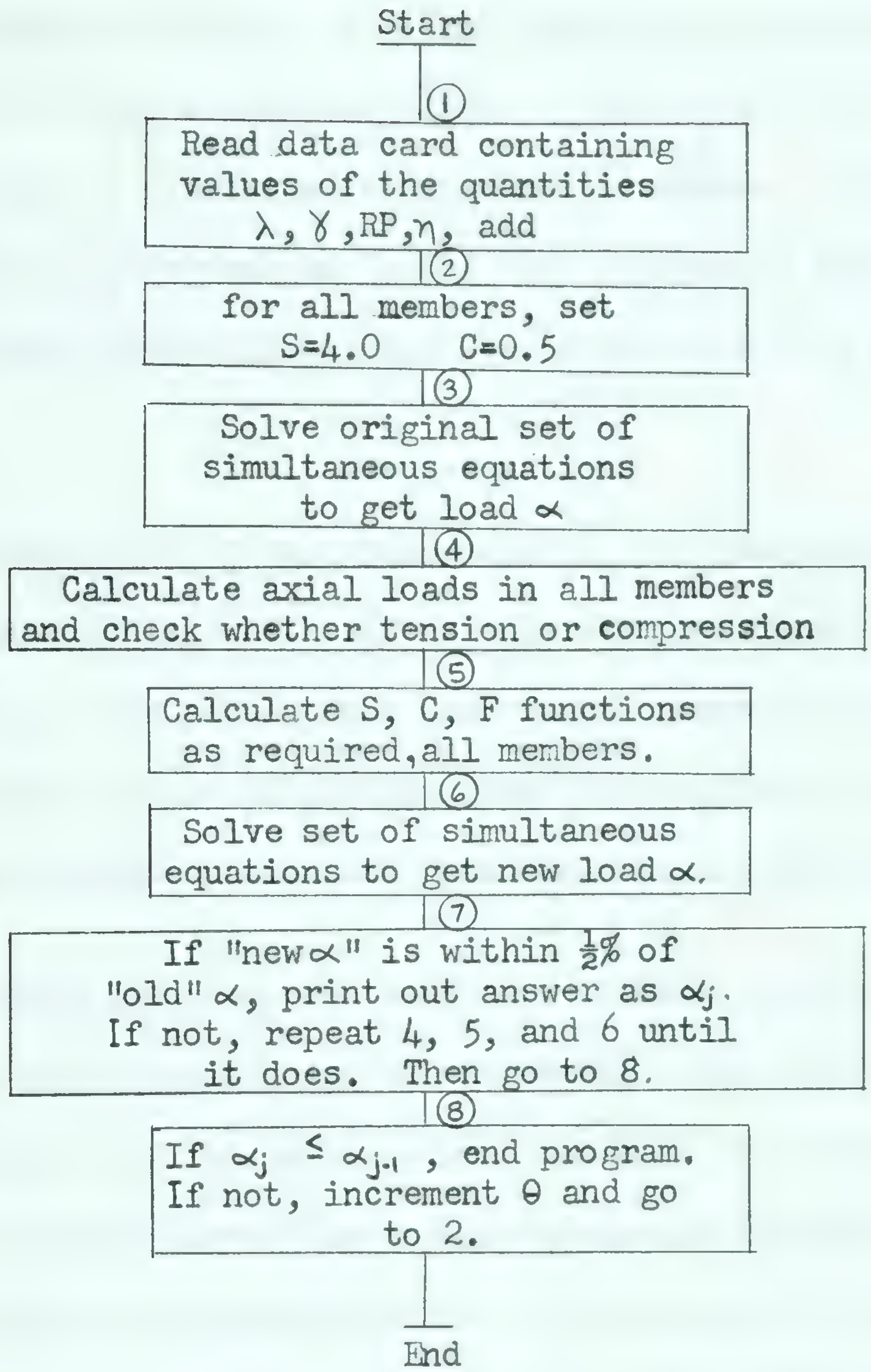


FIGURE D-2 Flow Diagram for Computer Program

The governing simultaneous equations were solved by using a determinant type of solution. If the number of equations is larger than three, it would be more expedient to use some other solution technique, such as a Gaussian elimination procedure. As an alternative to including this portion in the main program, a computer library program is sometimes available and can be called for by the main program.

In solving for the load parameter α, the program was instructed to repeat itself until the error in this value was less than one half of one percent. The use of this particular standard of accuracy was by choice nominal, but the results have indicated that this is a reasonable value since satisfactory load deflection curves were obtained.

The value of deflection used to increment each successive value of θ had a very direct bearing on the running time of the programs. For this reason, a relatively small θ increment was used initially in order to adequately define the rising portion of the load deflection curve, while for the flatter portion of the curve, the computer was instructed to increase the increment value. The initial values of the increment used for various values of RP, the ratio of the lateral to vertical load, are as follows:

RP	θ INCREMENT (RADIANS)
0.00	0.05
0.01	0.02
0.04	0.03
0.07	0.04
0.10	0.05

After four points were calculated, the computer doubled the increment, and after twelve points, the increment was made three times the initial value.

A preliminary program was run to indicate the ability of the computer to calculate accurate values of the S, C, and F functions throughout the range of axial load defined by the parameter $\phi = \sqrt{PL^2/EI}$ For a standard precision of eight significant digits in all calculations, very inconsistent results were evident in all functions for values of ϕ less than about twelve degrees. Knowing that at $\phi = 0$, $S = 4.0$, $C = 0.5$, and $F = 0.08\overline{3}$, and knowing also the calculated values of S, C, and F at $\phi = 12^\circ$, linear functions were used as replacements to the actual functions in the range $0<\phi<12^\circ$. The use of simple linear functions is justified since the actual functions exhibit such small percentage variation in this range (see figures 2-2,2-3,2-4) as compared to the remaining portions of the curves.

Before deciding on the use of the standard precision of eight significant digits for the running of the bulk of the programs, companion programs were run using a precision of twelve digits. Inaccuracies were found only in the fifth significant digit, this being considered trivial and unworthy of a refinement in precision past eight significant digits.

The programs proper are contained on the following pages. Input format is described on statement number 2 of each program. To run more than one data card, insert a ..I card and a ..branch 0040R

card before each successive data card. For the convenience of those wishing to investigate the details of these programs, or to modify them in any way, a list of Fortran II symbols used is given following the programs.

```
..I   ELASTIC STABILITY OF FRAMES
..I   FRAME WITH HINGED BASES - SMALL DEFLECTION THEORY
..LOAD FORTRAN EXECUTE
      READ 2,H,REI,RP,ADD
    2 FORMAT(F6.2,F6.2,F6.2,F6.2)
      G=H/REI
      PUNCH 4
    4 FORMAT(38H1 ELASTIC STABILITY OF FRAMES  PART 1B)
      PUNCH 3,H,G,RP,REI
    3 FORMAT(////,2X,7HLANDA= ,F5.2,5X,7HGAMMA= ,F5.2,5X,4HRP= ,F6.3,
     15X,5HREI= ,F6.2)
      DIMENSION P(50)
      P(1)=0.0
      E=2.718281828459
      PY=3.141592653590
      J=2
      ADDX=ADD
      TH=ADDX
    1 SINTH=SINF(TH)
      COSTH=COSF(TH)
      SAB=4.0
      SBC=4.0
      SCD=4.0
      CAB=0.5
      CBC=0.5
      CCD=0.5
      AA=3.*G+4.
      AB=2.
      AC=3.
      AD=2.
      AE=AA
      AF=AC
      AG=-((PY**2)*G*H)/12.
      AH=AG*(-1.)
      AI=(PY**2)*(RP+TH)
      YA=3.*G*TH
      YB=YA
      YC=6.*TH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THB=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THC=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P1=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AE*YA-AB*AD*YC-AF*YB*AA)/D
   10 RAB= P1/2.-(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H)
      PAB=PY*SQRTF(ABSF(RAB))
      IF(RAB)100,101,101
  100 IF(PAB-(12.*PY)/180.)14,14,12
   14 SAB=4.0+(PAB*(180.)*(0.0039965))/(12.*PY)
      CAB=0.5-(PAB*(180.)*(0.00110865))/(12.*PY)
      GO TO 20
   12 EP=E**PAB
      SIHPAB=0.5*(EP-(1./EP))
      COHPAB=0.5*(EP+(1./EP))
      SAB=(PAB*SIHPAB-(PAB**2)*COHPAB)/(2.*COHPAB-2.-PAB*SIHPAB)
      CAB=(PAB-SIHPAB)/(SIHPAB-PAB*COHPAB)
      GO TO 20
  101 IF(PAB-(12.*PY)/180.)15,15,13
   15 SAB=4.0-(PAB*(180.)*(0.0060669))/(12.*PY)
      CAB=0.5+(PAB*(180.)*(0.00110366))/(12.*PY)
      GO TO 20
   13 SINPAB=SINF(PAB)
```

```
      COSPAB=COSF(PAB)
      SAB=(PAB*SINPAB-(PAB**2)*COSPAB)/(2.-2.*COSPAB-PAB*SINPAB)
      CAB=(PAB-SINPAB)/(SINPAB-PAB*COSPAB)
      GO TO 20
   20 RBC=-(P1*G*H/2.+SBC*(1.+CBC)*(THB+THC)/(PY**2))*TH
     1     -(SCD*G*H*(1.-CCD**2)*(THC-TH)/(PY**2))
      PBC=PY*SQRTF(ABSF(RBC))
      IF(RBC)200,201,201
  200 IF(PBC-(12.*PY)/180.)24,24,22
   24 SBC=4.0+(PBC*(180.)*(0.0039965))/(12.*PY)
      CBC=0.5-(PBC*(180.)*(0.00110865))/(12.*PY)
      FBC=0.08333333-(PBC*(180.)*(0.00034900))/(12.*PY)
      GO TO 30
   22 EP=E**PBC
      EP2=E**(PBC/2.)
      SIHPBC=0.5*(EP-(1./EP))
      COHPBC=0.5*(EP+(1./EP))
      TAHPBC=(EP2-(1./EP2))/(EP2+(1./EP2))
      SBC=(PBC*SIHPBC-(PBC**2)*COHPBC)/(2.*COHPBC-2.-PBC*SIHPBC)
      CBC=(PBC-SIHPBC)/(SIHPBC-PBC*COHPBC)
      FBC=(1.-(2.*TAHPBC)/PBC)/(PBC**2)
      GO TO 30
  201 IF(PBC-(12.*PY)/180.)25,25,23
   25 SBC=4.0-(PBC*(180.)*(0.0060669))/(12.*PY)
      CBC=0.5+(PBC*(180.)*(0.00110366))/(12.*PY)
      FBC=0.08333333+(PBC*(180.)*(0.00036456))/(12.*PY)
      GO TO 30
   23 SINPBC=SINF(PBC)
      COSPBC=COSF(PBC)
      TANPBC=SINF(PBC/2.)/COSF(PBC/2.)
      SBC=(PBC*SINPBC-(PBC**2)*COSPBC)/(2.-2.*COSPBC-PBC*SINPBC)
      CBC=(PBC-SINPBC)/(SINPBC-PBC*COSPBC)
      FBC=((2.*TANPBC)/PBC-1.)/(PBC**2)
      GO TO 30
   30 RCD= P1/2.+(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H)
      PCD=PY*SQRTF(ABSF(RCD))
      IF(RCD)300,301,301
  300 IF(PCD-(12.*PY)/180.)34,34,32
   34 SCD=4.0+(PCD*(180.)*(0.0039965))/(12.*PY)
      CCD=0.5-(PCD*(180.)*(0.00110865))/(12.*PY)
      GO TO 40
   32 EP=E**PCD
      SIHPCD=0.5*(EP-(1./EP))
      COHPCD=0.5*(EP+(1./EP))
      SCD=(PCD*SIHPCD-(PCD**2)*COHPCD)/(2.*COHPCD-2.-PCD*SIHPCD)
      CCD=(PCD-SIHPCD)/(SIHPCD-PCD*COHPCD)
      GO TO 40
  301 IF(PCD-(12.*PY)/180.)35,35,33
   35 SCD=4.0-(PCD*(180.)*(0.0060669))/(12.*PY)
      CCD=0.5+(PCD*(180.)*(0.00110366))/(12.*PY)
      GO TO 40
   33 SINPCD=SINF(PCD)
      COSPCD=COSF(PCD)
      SCD=(PCD*SINPCD-(PCD**2)*COSPCD)/(2.-2.*COSPCD-PCD*SINPCD)
      CCD=(PCD-SINPCD)/(SINPCD-PCD*COSPCD)
      GO TO 40
   40 AA=SAB*G*(1.-(CAB**2))+SBC
      AB=SBC*CBC
      AC=SAB*(1.-(CAB**2))
      AD=AB
```

```
      AE=SCD*G*(1.-(CCD**2))+SBC
      AF=SCD*(1.-(CCD**2))
      AG=-(SBC*(1.-CBC)*G*H*(PY**2)*FBC)/2.
      AH=AG*(-1.)
      AI=(PY**2)*(RP+TH)
      YA=SAB*G*(1.-(CAB**2))*TH
      YB=SCD*G*(1.-(CCD**2))*TH
      YC=(SAB*(1.-(CAB**2))+SCD*(1.-(CCD**2)))*TH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THB=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THC=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P2=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AE*YA-AB*AD*YC-AF*YB*AA)/D
      PPP=(ABSF(P2-P1))/P1-0.005
      IF(PPP)51,51,50
   50 P1=P2
      GO TO 10
   51 P(J)=P2
      THD=(1.+CCD)*TH-CCD*THC
      PUNCH 52,TH,P2,THB,THC,THD
   52 FORMAT(//,2X, 7HTHETA= ,F11.8,9H  ALPHA= ,F10.6,7H  THB= ,F6.3,
     17H  THC= ,F6.3,7H  THD= ,F6.3)
      PUNCH 53,RAB,RBC,RCD
   53 FORMAT(41X,5HRAB= ,F6.3,7H  RBC= ,F6.3,7H  RCD= ,F6.3)
      PPPP=P(J)-P(J-1)
      IF(PPPP)70,70,60
   60 IF(J-5)69,63,64
   63 ADDX=2.*ADD
      GO TO 69
   64 IF(J-13)69,65,69
   65 ADDX=3.*ADD
      GO TO 69
   69 J=J+1
      TH=TH+ADDX
      IF(TH-1.05)88,88,70
   88 GO TO 1
   70 CALL EXIT
   62 END
```

```
..I   ELASTIC STABILITY OF FRAMES
..I   FRAME WITH HINGED BASES - LARGE DEFLECTION THEORY
..LOAD FORTRAN EXECUTE
      READ 2,H,REI,RP,ADD
    2 FORMAT(F6.2,F6.2,F6.2,F6.2)
      G=H/REI
      PUNCH 4
    4 FORMAT(38H1 ELASTIC STABILITY OF FRAMES  PART 1A)
      PUNCH 3,H,G,RP,REI
    3 FORMAT(////,2X,7HLANDA= ,F5.2,5X,7HGAMMA= ,F5.2,5X,4HRP= ,F6.3,
     15X,5HREI= ,F6.2)
      DIMENSION P(50)
      P(1)=0.0
      E=2.718281828459
      PY=3.141592653590
      J=2
      ADDX=ADD
      TH=ADDX
    1 SINTH=SINF(TH)
      COSTH=COSF(TH)
      TANTH=SINTH/COSTH
      SAB=4.0
      SBC=4.0
      SCD=4.0
      CAB=0.5
      CBC=0.5
      CCD=0.5
      AA=3.*G+4.
      AB=2.
      AC=3.
      AD=2.
      AE=AA
      AF=AC
      AG=-((PY**2)*G*H)/12.
      AH=AG*(-1.)
      AI=(PY**2)*(RP*COSTH+SINTH)
      YA=3.*G*TANTH
      YB=YA
      YC=6.*TANTH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THB=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THC=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P1=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AE*YA-AB*AD*YC-AF*YB*AA)/D
   10 RAB=(P1/2.-(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H))/COSTH
     1     +(SAB*(1.-CAB**2)*(THB-TANTH)*TANTH)/(PY**2)
      PAB=PY*SQRTF(ABSF(RAB))
      IF(RAB)100,101,101
  100 IF(PAB-(12.*PY)/180.)14,14,12
   14 SAB=4.0+(PAB*(180.)*(0.0039965))/(12.*PY)
      CAB=0.5-(PAB*(180.)*(0.00110865))/(12.*PY)
      GO TO 20
   12 EP=E**PAB
      SIHPAB=0.5*(EP-(1./EP))
      COHPAB=0.5*(EP+(1./EP))
      SAB=(PAB*SIHPAB-(PAB**2)*COHPAB)/(2.*COHPAB-2.-PAB*SIHPAB)
      CAB=(PAB-SIHPAB)/(SIHPAB-PAB*COHPAB)
      GO TO 20
  101 IF(PAB-(12.*PY)/180.)15,15,13
   15 SAB=4.0-(PAB*(180.)*(0.0060669))/(12.*PY)
      CAB=0.5+(PAB*(180.)*(0.00110366))/(12.*PY)
```

```
      GO TO 20
   13 SINPAB=SINF(PAB)
      COSPAB=COSF(PAB)
      SAB=(PAB*SINPAB-(PAB**2)*COSPAB)/(2.-2.*COSPAB-PAB*SINPAB)
      CAB=(PAB-SINPAB)/(SINPAB-PAB*COSPAB)
      GO TO 20
   20 RBC=-(P1*G*H/2.+SBC*(1.+CBC)*(THB+THC)/(PY**2))*TANTH
     1    -(SCD*G*H*(1.-CCD**2)*(THC-TANTH)/(PY**2))/COSTH
      PBC=PY*SQRTF(ABSF(RBC))
      IF(PBC)200,201,201
  200 IF(PBC-(12.*PY)/180.)24,24,22
   24 SBC=4.0+(PBC*(180.)*(0.0039965))/(12.*PY)
      CBC=0.5-(PBC*(180.)*(0.00110865))/(12.*PY)
      FBC=0.08333333-(PBC*(180.)*(0.00034900))/(12.*PY)
      GO TO 30
   22 EP=E**PBC
      EP2=E**(PBC/2.)
      SIHPBC=0.5*(EP-(1./EP))
      COHPBC=0.5*(EP+(1./EP))
      TAHPBC=(EP2-(1./EP2))/(EP2+(1./EP2))
      SBC=(PBC*SIHPBC-(PBC**2)*COHPBC)/(2.*COHPBC-2.-PBC*SIHPBC)
      CBC=(PBC-SIHPBC)/(SIHPBC-PBC*COHPBC)
      FBC=(1.-(2.*TAHPBC)/PBC)/(PBC**2)
      GO TO 30
  201 IF(PBC-(12.*PY)/180.)25,25,23
   25 SBC=4.0-(PBC*(180.)*(0.0060669))/(12.*PY)
      CBC=0.5+(PBC*(180.)*(0.00110366))/(12.*PY)
      FBC=0.08333333+(PBC*(180.)*(0.00036456))/(12.*PY)
      GO TO 30
   23 SINPBC=SINF(PBC)
      COSPBC=COSF(PBC)
      TANPBC=SINF(PBC/2.)/COSF(PBC/2.)
      SBC=(PBC*SINPBC-(PBC**2)*COSPBC)/(2.-2.*COSPBC-PBC*SINPBC)
      CBC=(PBC-SINPBC)/(SINPBC-PBC*COSPBC)
      FBC=((2.*TANPBC)/PBC-1.)/(PBC**2)
      GO TO 30
   30 RCD=(P1/2.+(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H))/COSTH
     1    +(SCD*(1.-CCD**2)*(THC-TANTH)*TANTH)/(PY**2)
      PCD=PY*SQRTF(ABSF(RCD))
      IF(RCD)300,301,301
  300 IF(PCD-(12.*PY)/180.)34,34,32
   34 SCD=4.0+(PCD*(180.)*(0.0039965))/(12.*PY)
      CCD=0.5-(PCD*(180.)*(0.00110865))/(12.*PY)
      GO TO 40
   32 EP=E**PCD
      SIHPCD=0.5*(EP-(1./EP))
      COHPCD=0.5*(EP+(1./EP))
      SCD=(PCD*SIHPCD-(PCD**2)*COHPCD)/(2.*COHPCD-2.-PCD*SIHPCD)
      CCD=(PCD-SIHPCD)/(SIHPCD-PCD*COHPCD)
      GO TO 40
  301 IF(PCD-(12.*PY)/180.)35,35,33
   35 SCD=4.0-(PCD*(180.)*(0.0060669))/(12.*PY)
      CCD=0.5+(PCD*(180.)*(0.00110366))/(12.*PY)
      GO TO 40
   33 SINPCD=SINF(PCD)
      COSPCD=COSF(PCD)
      SCD=(PCD*SINPCD-(PCD**2)*COSPCD)/(2.-2.*COSPCD-PCD*SINPCD)
      CCD=(PCD-SINPCD)/(SINPCD-PCD*COSPCD)
      GO TO 40
   40 AA=SAB*G*(1.-(CAB**2))+SBC
```

```
      AB=SBC*CBC
      AC=SAB*(1.-(CAB**2))
      AD=AB
      AE=SCD*G*(1.-(CCD**2))+SBC
      AF=SCD*(1.-(CCD**2))
      AG=-(SBC*(1.-CBC)*G*H*(PY**2)*FBC)/2.
      AH=AG*(-1.)
      AI=(PY**2)*(RP*COSTH+SINTH)
      YA=SAB*G*(1.-(CAB**2))*TANTH
      YB=SCD*G*(1.-(CCD**2))*TANTH
      YC=(SAB*(1.-(CAB**2))+SCD*(1.-(CCD**2)))*TANTH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THB=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THC=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P2=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AE*YA-AB*AD*YC-AF*YB*AA)/D
      PPP=(ABSF(P2-P1))/P1-0.005
      IF(PPP)51,51,50
   50 P1=P2
      GO TO 10
   51 P(J)=P2
      THD=(1.+CCD)*TANTH-CCD*THC
      IF(ABSF(P2)-0.6)49,49,70
   49 PUNCH 52,TH,P2,THB,THC,THD
   52 FORMAT(//,2X, 7HTHETA= ,F11.8,9H  ALPHA= ,F10.6,7H  THB= ,F6.3,
     17H  THC= ,F6.3,7H  THD= ,F6.3)
      PUNCH 53,RAB,RBC,RCD
   53 FORMAT(41X,5HRAB= ,F6.3,7H  RBC= ,F6.3,7H  RCD= ,F6.3)
      PPPP=P(J)-P(J-1)
      IF(PPPP)70,70,60
   60 IF(J-5)69,63,64
   63 ADDX=2.*ADD
      GO TO 69
   64 IF(J-13)69,65,69
   65 ADDX=3.*ADD
      GO TO 69
   69 J=J+1
      TH=TH+ADDX
      IF(TH-0.98)88,88,70
   88 GO TO 1
   70 CALL EXIT
   62 END
```

```
..I   ELASTIC STABILITY OF FRAMES
..I   FRAME WITH FIXED BASES - SMALL DEFLECTION THEORY
..LOAD FORTRAN EXECUTE
      READ 2,H,REI,RP,ADD
    2 FORMAT(F6.2,F6.2,F6.2,F6.2)
      G=H/REI
      PUNCH 4
    4 FORMAT(38H1 ELASTIC STABILITY OF FRAMES  PART 2B)
      PUNCH 3,H,G,RP,REI
    3 FORMAT(////,2X,7HLANDA= ,F5.2,5X,7HGAMMA= ,F5.2,5X,4HRP= ,F6.3,
     15X,5HREI= ,F6.2)
      DIMENSION P(50)
      P(1)=0.0
      E=2.718281828459
      PY=3.141592653590
      J=2
      ADDX=ADD
      TH=ADDX
    1 SINTH=SINF(TH)
      COSTH=COSF(TH)
      SAB=4.0
      SBC=4.0
      SCD=4.0
      CAB=0.5
      CBC=0.5
      CCD=0.5
      AA=4.*G+4.
      AB=2.
      AC=6.
      AD=2.
      AE=AA
      AF=AC
      AG=-((PY**2)*G*H)/12.
      AH=AG*(-1.)
      AI=(PY**2)*(RP+TH)
      YA=6.*G*TH
      YB=YA
      YC=24.*TH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THB=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THC=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P1=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AF*YA-AB*AD*YC-AF*YB*AA)/D
   10 RAB= P1/2.-(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H)
      PAB=PY*SQRTF(ABSF(RAB))
      IF(RAB)100,101,101
  100 IF(PAB-(12.*PY)/180.)14,14,12
   14 SAB=4.0+(PAB*(180.)*(0.0039965))/(12.*PY)
      CAB=0.5-(PAB*(180.)*(0.00110865))/(12.*PY)
      GO TO 20
   12 EP=E**PAB
      SIHPAB=0.5*(EP-(1./EP))
      COHPAB=0.5*(EP+(1./EP))
      SAB=(PAB*SIHPAB-(PAB**2)*COHPAB)/(2.*COHPAB-2.-PAB*SIHPAB)
      CAB=(PAB-SIHPAB)/(SIHPAB-PAB*COHPAB)
      GO TO 20
  101 IF(PAB-(12.*PY)/180.)15,15,13
   15 SAB=4.0-(PAB*(180.)*(0.0060669))/(12.*PY)
      CAB=0.5+(PAB*(180.)*(0.00110366))/(12.*PY)
      GO TO 20
   13 SINPAB=SINF(PAB)
```

```
      COSPAB=COSF(PAB)
      SAB=(PAB*SINPAB-(PAB**2)*COSPAB)/(2.-2.*COSPAB-PAB*SINPAB)
      CAB=(PAB-SINPAB)/(SINPAB-PAB*COSPAB)
      GO TO 20
   20 RBC=-(P1*G*H/2.+SBC*(1.+CBC)*(THB+THC)/(PY**2))*TH
     1      -(SCD*G*H*(1.+CCD)*(THC-2.*TH)/(PY**2))
      PBC=PY*SQRTF(ABSF(RBC))
      IF(RBC)200,201,201
  200 IF(PBC-(12.*PY)/180.)24,24,22
   24 SBC=4.0+(PBC*(180.)*(0.0039965))/(12.*PY)
      CBC=0.5-(PBC*(180.)*(0.00110865))/(12.*PY)
      FBC=0.08333333-(PBC*(180.)*(0.00034900))/(12.*PY)
      GO TO 30
   22 EP=E**PBC
      EP2=E**(PBC/2.)
      SIHPBC=0.5*(EP-(1./EP))
      COHPBC=0.5*(EP+(1./EP))
      TAHPBC=(EP2-(1./EP2))/(EP2+(1./EP2))
      SBC=(PBC*SIHPBC-(PBC**2)*COHPBC)/(2.*COHPBC-2.-PBC*SIHPBC)
      CBC=(PBC-SIHPBC)/(SIHPBC-PBC*COHPBC)
      FBC=(1.-(2.*TAHPBC)/PBC)/(PBC**2)
      GO TO 30
  201 IF(PBC-(12.*PY)/180.)25,25,23
   25 SBC=4.0-(PBC*(180.)*(0.0060669))/(12.*PY)
      CBC=0.5+(PBC*(180.)*(0.00110366))/(12.*PY)
      FBC=0.08333333+(PBC*(180.)*(0.00036456))/(12.*PY)
      GO TO 30
   23 SINPBC=SINF(PBC)
      COSPBC=COSF(PBC)
      TANPBC=SINF(PBC/2.)/COSF(PBC/2.)
      SBC=(PBC*SINPBC-(PBC**2)*COSPBC)/(2.-2.*COSPBC-PBC*SINPBC)
      CBC=(PBC-SINPBC)/(SINPBC-PBC*COSPBC)
      FBC=((2.*TANPBC)/PBC-1.)/(PBC**2)
      GO TO 30
   30 RCD= P1/2.+(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H)
      PCD=PY*SQRTF(ABSF(RCD))
      IF(RCD)300,301,301
  300 IF(PCD-(12.*PY)/180.)34,34,32
   34 SCD=4.0+(PCD*(180.)*(0.0039965))/(12.*PY)
      CCD=0.5-(PCD*(180.)*(0.00110865))/(12.*PY)
      GO TO 40
   32 EP=E**PCD
      SIHPCD=0.5*(EP-(1./EP))
      COHPCD=0.5*(EP+(1./EP))
      SCD=(PCD*SIHPCD-(PCD**2)*COHPCD)/(2.*COHPCD-2.-PCD*SIHPCD)
      CCD=(PCD-SIHPCD)/(SIHPCD-PCD*COHPCD)
      GO TO 40
  301 IF(PCD-(12.*PY)/180.)35,35,33
   35 SCD=4.0-(PCD*(180.)*(0.0060669))/(12.*PY)
      CCD=0.5+(PCD*(180.)*(0.00110366))/(12.*PY)
      GO TO 40
   33 SINPCD=SINF(PCD)
      COSPCD=COSF(PCD)
      SCD=(PCD*SINPCD-(PCD**2)*COSPCD)/(2.-2.*COSPCD-PCD*SINPCD)
      CCD=(PCD-SINPCD)/(SINPCD-PCD*COSPCD)
      GO TO 40
   40 AA=SAB*G+SBC
      AB=SBC*CBC
      AC=SAB*(1.+CAB)
      AD=AB
```

```
      AE=SCD*G+SBC
      AF=SCD*(1.+CCD)
      AG=-(SBC*(1.-CBC)*G*H*(PY**2)*FBC)/2.
      AH=AG*(-1.)
      AI=(PY**2)*(RP+TH)
      YA=SAB*G*(1.+CAB)*TH
      YB=SCD*G*(1.+CCD)*TH
      YC=(2.*SAB*(1.+CAB)+2.*SCD*(1.+CCD))*TH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THB=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THC=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P2=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AE*YA-AB*AD*YC-AF*YB*AA)/D
      PPP=(ABSF(P2-P1))/P1-0.005
      IF(PPP)51,51,50
   50 P1=P2
      GO TO 10
   51 P(J)=P2
      PUNCH 52,TH,P2,THB,THC
   52 FORMAT(//,2X, 7HTHETA= ,F11.8,9H  ALPHA= ,F10.6,7H  THB= ,F6.3,
     17H  THC= ,F6.3)
      PUNCH 53,RAB,RBC,RCD
   53 FORMAT(41X,5HRAB= ,F6.3,7H  RBC= ,F6.3,7H  RCD= ,F6.3)
      PPPP=P(J)-P(J-1)
      IF(PPPP)70,70,60
   60 IF(J-5)69,63,64
   63 ADDX=2.*ADD
      GO TO 69
   64 IF(J-13)69,65,69
   65 ADDX=3.*ADD
      GO TO 69
   69 J=J+1
      TH=TH+ADDX
      IF(TH-1.05)88,88,70
   88 GO TO 1
   70 CALL EXIT
   62 END
```

```
..I   ELASTIC STABILITY OF FRAMES
..I   FRAME WITH FIXED BASES - LARGE DEFLECTION THEORY
..LOAD FORTRAN EXECUTE
      READ 2,H,REI,RP,ADD
    2 FORMAT(F6.2,F6.2,F6.2,F6.2)
      G=H/REI
      PUNCH 4
    4 FORMAT(38H1 ELASTIC STABILITY OF FRAMES  PART 2A)
      PUNCH 3,H,G,RP,REI
    3 FORMAT(////,2X,7HLANDA= ,F5.2,5X,7HGAMMA= ,F5.2,5X,4HRP= ,F6.3,
     15X,5HREI= ,F6.2)
      DIMENSION P(50)
      P(1)=0.0
      E=2.718281828459
      PY=3.141592653590
      J=2
      ADDX=ADD
      TH=ADDX
    1 SINTH=SINF(TH)
      COSTH=COSF(TH)
      TANTH=SINTH/COSTH
      SAB=4.0
      SBC=4.0
      SCD=4.0
      CAB=0.5
      CBC=0.5
      CCD=0.5
      AA=4.*G+4.
      AB=2.
      AC=6.
      AD=2.
      AE=AA
      AF=AC
      AG=-((PY**2)*G*H)/12.
      AH=AG*(-1.)
      AI=(PY**2)*(RP*COSTH+SINTH)
      YA=6.*G*TANTH
      YB=YA
      YC=24.*TANTH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THB=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THC=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P1=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AE*YA-AB*AD*YC-AE*YB*AA)/D
   10 RAB=(P1/2.-(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H))/COSTH
     1  +(SAB*(1.+CAB)*(THB-2.*TANTH)*TANTH/(PY**2))
      PAB=PY*SQRTF(ABSF(RAB))
      IF(RAB)100,101,101
  100 IF(PAB-(12.*PY)/180.)14,14,12
   14 SAB=4.0+(PAB*(180.)*(0.0039965))/(12.*PY)
      CAB=0.5-(PAB*(180.)*(0.00110865))/(12.*PY)
      GO TO 20
   12 EP=E**PAB
      SIHPAB=0.5*(EP-(1./EP))
      COHPAB=0.5*(EP+(1./EP))
      SAB=(PAB*SIHPAB-(PAB**2)*COHPAB)/(2.*COHPAB-2.-PAB*SIHPAB)
      CAB=(PAB-SIHPAB)/(SIHPAB-PAB*COHPAB)
      GO TO 20
  101 IF(PAB-(12.*PY)/180.)15,15,13
   15 SAB=4.0-(PAB*(180.)*(0.0060669))/(12.*PY)
      CAB=0.5+(PAB*(180.)*(0.00110366))/(12.*PY)
```

```
      GO TO 20
   13 SINPAB=SINF(PAB)
      COSPAB=COSF(PAB)
      SAB=(PAB*SINPAB-(PAB**2)*COSPAB)/(2.-2.*COSPAB-PAB*SINPAB)
      CAB=(PAB-SINPAB)/(SINPAB-PAB*COSPAB)
      GO TO 20
   20 RBC=-(PI*G*H/2.+SBC*(1.+CBC)*(THB+THC)/(PY**2))*TANTH
     1  -(SCD*G*H*(1.+CCD)*(THC-2.*TANTH)/(PY**2))/COSTH
      PBC=PY*SQRTF(ABSF(RBC))
      IF(RBC)200,201,201
  200 IF(PBC-(12.*PY)/180.)24,24,22
   24 SBC=4.0+(PBC*(180.)*(0.0039965))/(12.*PY)
      CBC=0.5-(PBC*(180.)*(0.00110865))/(12.*PY)
      FBC=0.08333333-(PBC*(180.)*(0.00034900))/(12.*PY)
      GO TO 30
   22 EP=E**PBC
      EP2=E**(PBC/2.)
      SIHPBC=0.5*(EP-(1./EP))
      COHPBC=0.5*(EP+(1./EP))
      TAHPBC=(EP2-(1./EP2))/(EP2+(1./EP2))
      SBC=(PBC*SIHPBC-(PBC**2)*COHPBC)/(2.*COHPBC-2.-PBC*SIHPBC)
      CBC=(PBC-SIHPBC)/(SIHPBC-PBC*COHPBC)
      FBC=(1.-(2.*TAHPBC)/PBC)/(PBC**2)
      GO TO 30
  201 IF(PBC-(12.*PY)/180.)25,25,23
   25 SBC=4.0-(PBC*(180.)*(0.0060669))/(12.*PY)
      CBC=0.5+(PBC*(180.)*(0.00110366))/(12.*PY)
      FBC=0.08333333+(PBC*(180.)*(0.00036456))/(12.*PY)
      GO TO 30
   23 SINPBC=SINF(PBC)
      COSPBC=COSF(PBC)
      TANPBC=SINF(PBC/2.)/COSF(PBC/2.)
      SBC=(PBC*SINPBC-(PBC**2)*COSPBC)/(2.-2.*COSPBC-PBC*SINPBC)
      CBC=(PBC-SINPBC)/(SINPBC-PBC*COSPBC)
      FBC=((2.*TANPBC)/PBC-1.)/(PBC**2)
      GO TO 30
   30 RCD=(PI/2.+(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H))/COSTH
     1  +(SCD*(1.+CCD)*(THC-2.*TANTH)*TANTH/(PY**2))
      PCD=PY*SQRTF(ABSF(RCD))
      IF(RCD)300,301,301
  300 IF(PCD-(12.*PY)/180.)34,34,32
   34 SCD=4.0+(PCD*(180.)*(0.0039965))/(12.*PY)
      CCD=0.5-(PCD*(180.)*(0.00110865))/(12.*PY)
      GO TO 40
   32 EP=E**PCD
      SIHPCD=0.5*(EP-(1./EP))
      COHPCD=0.5*(EP+(1./EP))
      SCD=(PCD*SIHPCD-(PCD**2)*COHPCD)/(2.*COHPCD-2.-PCD*SIHPCD)
      CCD=(PCD-SIHPCD)/(SIHPCD-PCD*COHPCD)
      GO TO 40
  301 IF(PCD-(12.*PY)/180.)35,35,33
   35 SCD=4.0-(PCD*(180.)*(0.0060669))/(12.*PY)
      CCD=0.5+(PCD*(180.)*(0.00110366))/(12.*PY)
      GO TO 40
   33 SINPCD=SINF(PCD)
      COSPCD=COSF(PCD)
      SCD=(PCD*SINPCD-(PCD**2)*COSPCD)/(2.-2.*COSPCD-PCD*SINPCD)
      CCD=(PCD-SINPCD)/(SINPCD-PCD*COSPCD)
      GO TO 40
   40 AA=SAB*G+SBC
```

```
      AB=SBC*CBC
      AC=SAB*(1.+CAB)
      AD=AB
      AE=SCD*G+SBC
      AF=SCD*(1.+CCD)
      AG=-(SBC*(1.-CBC)*G*H*(PY**2)*FBC)/2.
      AH=AG*(-1.)
      AI=(PY**2)*(RP*COSTH+SINTH)
      YA=SAB*G*(1.+CAB)*TANTH
      YB=SCD*G*(1.+CCD)*TANTH
      YC=(2.*SAB*(1.+CAB)+2.*SCD*(1.+CCD))*TANTH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THB=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THC=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P2=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AE*YA-AB*AD*YC-AF*YB*AA)/D
      PPP=(ABSF(P2-P1))/P1-0.005
      IF(PPP)51,51,50
   50 P1=P2
      GO TO 10
   51 P(J)=P2
      IF(ABSF(P2)-2.4)49,49,70
   49 PUNCH 52,TH,P2,THB,THC
   52 FORMAT(//,2X, 7HTHETA= ,F11.8,9H  ALPHA= ,F10.6,7H  THB= ,F6.3,
     17H  THC= ,F6.3)
      PUNCH 53,RAB,RBC,RCD
   53 FORMAT(41X,5HRAB= ,F6.3,7H  RBC= ,F6.3,7H  RCD= ,F6.3)
      PPPP=P(J)-P(J-1)
      IF(PPPP)70,70,60
   60 IF(J-5)69,63,64
   63 ADDX=2.*ADD
      GO TO 69
   64 IF(J-13)69,65,69
   65 ADDX=3.*ADD
      GO TO 69
   69 J=J+1
      TH=TH+ADDX
      IF(TH-1.05)88,88,70
   88 GO TO 1
   70 CALL EXIT
   62 END
```

```
..I   ELASTIC STABILITY OF FRAMES
..I   FRAME WITH PARTIAL BASE FIXITY  (SMALL DEFLECTION THEORY)
..LOAD FORTRAN EXECUTE
      READ 2,H,REI,RP,ADD,F
    2 FORMAT(F6.2,F6.2,F6.2,F6.2,F6.2)
      G=H/REI
      PUNCH 4
    4 FORMAT(38H1 ELASTIC STABILITY OF FRAMES  PART 3B)
      PUNCH 3,H,G,RP,REI,F
    3 FORMAT(////,2X,7HLANDA= ,F5.2,3X,7HGAMMA= ,F5.2,3X,4HRP= ,F6.3,
     13X,5HREI= ,F6.2,3X,5HETA= ,F6.2)
      DIMENSION P(50)
      P(1)=0.0
      E=2.718281828459
      PY=3.141592653590
      J=2
      ADDX=ADD
      TH=ADDX
    1 SINTH=SINF(TH)
      COSTH=COSF(TH)
      SAB=4.0
      SBC=4.0
      SCD=4.0
      CAB=0.5
      CBC=0.5
      CCD=0.5
      AA=-6.*G-8.*F-8.-10.*F/G
      AB=-4.-4.*F-8.*F/G
      AC=-5.-18.*F/G
      AD=AB
      AE=AA
      AF=AC
      AG=-((PY**2)*G*H)/12.
      AH=AG*(-1.)
      AI=(PY**2)*(RP+TH)
      YA=(-6.*G-18.)*TH
      YB=YA
      YC=-12.*TH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THA=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THD=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P1=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AE*YA-AB*AD*YC-AF*YB*AA)/D
      THB=3.*TH-(4.*G+4.*F)*THA/(2.*G)-2.*F*THD/(2.*G)
      THC=3.*TH-(4.*G+4.*F)*THD/(2.*G)-2.*F*THA/(2.*G)
   10 RAB= P1/2.-(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H)
      PAB=PY*SQRTF(ABSF(RAB))
      IF(RAB)100,101,101
  100 IF(PAB-(12.*PY)/180.)14,14,12
   14 SAB=4.0+(PAB*(180.)*(0.0039965))/(12.*PY)
      CAB=0.5-(PAB*(180.)*(0.00110865))/(12.*PY)
      GO TO 20
   12 EP=E**PAB
      SIHPAB=0.5*(EP-(1./EP))
      COHPAB=0.5*(EP+(1./EP))
      SAB=(PAB*SIHPAB-(PAB**2)*COHPAB)/(2.*COHPAB-2.-PAB*SIHPAB)
      CAB=(PAB-SIHPAB)/(SIHPAB-PAB*COHPAB)
      GO TO 20
  101 IF(PAB-(12.*PY)/180.)15,15,13
   15 SAB=4.0-(PAB*(180.)*(0.0060669))/(12.*PY)
      CAB=0.5+(PAB*(180.)*(0.00110366))/(12.*PY)
```

```
      GO TO 20
   13 SINPAB=SINF(PAB)
      COSPAB=COSF(PAB)
      SAB=(PAB*SINPAB-(PAB**2)*COSPAB)/(2.-2.*COSPAB-PAB*SINPAB)
      CAB=(PAB-SINPAB)/(SINPAB-PAB*COSPAB)
      GO TO 20
   20 RBC=-(P1*G*H/2.+SBC*(1.+CBC)*(THB+THC)/(PY**2))*TH
     1 -(SCD*G*H*(1.+CCD)*(THC+THD-2.*TH))/(PY**2)
      PBC=PY*SQRTF(ABSF(RBC))
      IF(RBC)200,201,201
  200 IF(PBC-(12.*PY)/180.)24,24,22
   24 SBC=4.0+(PBC*(180.)*(0.0039965))/(12.*PY)
      CBC=0.5-(PBC*(180.)*(0.00110865))/(12.*PY)
      FBC=0.08333333-(PBC*(180.)*(0.00034900))/(12.*PY)
      GO TO 30
   22 EP=E**PBC
      EP2=E**(PBC/2.)
      SIHPBC=0.5*(EP-(1./EP))
      COHPBC=0.5*(EP+(1./EP))
      TAHPBC=(EP2-(1./EP2))/(EP2+(1./EP2))
      SBC=(PBC*SIHPBC-(PBC**2)*COHPBC)/(2.*COHPBC-2.-PBC*SIHPBC)
      CBC=(PBC-SIHPBC)/(SIHPBC-PBC*COHPBC)
      FBC=(1.-(2.*TAHPBC)/PBC)/(PBC**2)
      GO TO 30
  201 IF(PBC-(12.*PY)/180.)25,25,23
   25 SBC=4.0-(PBC*(180.)*(0.0060669))/(12.*PY)
      CBC=0.5+(PBC*(180.)*(0.00110366))/(12.*PY)
      FBC=0.08333333+(PBC*(180.)*(0.00036456))/(12.*PY)
      GO TO 30
   23 SINPBC=SINF(PBC)
      COSPBC=COSF(PBC)
      TANPBC=SINF(PBC/2.)/COSF(PBC/2.)
      SBC=(PBC*SINPBC-(PBC**2)*COSPBC)/(2.-2.*COSPBC-PBC*SINPBC)
      CBC=(PBC-SINPBC)/(SINPBC-PBC*COSPBC)
      FBC=((2.*TANPBC)/PBC-1.)/(PBC**2)
      GO TO 30
   30 RCD= P1/2.+(SBC*(1.+CBC)*(THB+THC))/((PY**2)*G*H)
      PCD=PY*SQRTF(ABSF(RCD))
      IF(RCD)300,301,301
  300 IF(PCD-(12.*PY)/180.)34,34,32
   34 SCD=4.0+(PCD*(180.)*(0.0039965))/(12.*PY)
      CCD=0.5-(PCD*(180.)*(0.00110865))/(12.*PY)
      GO TO 40
   32 EP=E**PCD
      SIHPCD=0.5*(EP-(1./EP))
      COHPCD=0.5*(EP+(1./EP))
      SCD=(PCD*SIHPCD-(PCD**2)*COHPCD)/(2.*COHPCD-2.-PCD*SIHPCD)
      CCD=(PCD-SIHPCD)/(SIHPCD-PCD*COHPCD)
      GO TO 40
  301 IF(PCD-(12.*PY)/180.)35,35,33
   35 SCD=4.0-(PCD*(180.)*(0.0060669))/(12.*PY)
      CCD=0.5+(PCD*(180.)*(0.00110366))/(12.*PY)
      GO TO 40
   33 SINPCD=SINF(PCD)
      COSPCD=COSF(PCD)
      SCD=(PCD*SINPCD-(PCD**2)*COSPCD)/(2.-2.*COSPCD-PCD*SINPCD)
      CCD=(PCD-SINPCD)/(SINPCD-PCD*COSPCD)
      GO TO 40
   40 AA=SAB*G*CAB-(SAB*G+SBC)*(SAB*G+4.*F)/(SAB*G*CAB)
     1 -2.*F*SBC*CBC/(SCD*G*CCD)
```

```
      AB=-SBC*CBC*(SAB*G+4.*F)/(SAB*G*CAB)
     1 -(SBC+SCD*G)*(2.*F)/(SCD*G*CCD)
      AC=SAB*(1.+CAB)-(1.+CAB)*(SAB*G+4.*F)/(G*CAB)
     1 -(1.+CCD)*(2.*F)/(G*CCD)
      AD=-(SAB*G+SBC)*(2.*F)/(SAB*G*CAB)
     1 -SBC*CBC*(SCD*G+4.*F)/(SCD*G*CCD)
      AE=SCD*G*CCD-(SBC+SCD*G)*(SCD*G+4.*F)/(SCD*G*CCD)
     1 -SBC*CBC*(2.*F)/(SAB*G*CAB)
      AF=SCD*(1.+CCD)-(1.+CCD)*(SCD*G+4.*F)/(G*CCD)
     1 -(1.+CAB)*(2.*F)/(G*CAB)
      AG=-SBC*(1.-CBC)*G*H*(PY**2)*FBC/2.
      AH=AG*(-1.)
      AI=(PY**2)*(RP+TH)
      YA=(SAB*G*(1.+CAB)-(SAB*G+SBC)*(1.+CAB)/CAB
     1 -SBC*CBC*(1.+CCD)/CCD)*TH
      YB=(SCD*G*(1.+CCD)-(SBC+SCD*G)*(1.+CCD)/CCD
     1 -SBC*CBC*(1.+CAB)/CAB)*TH
      YC=(2.*SAB*(1.+CAB)+2.*SCD*(1.+CCD)-SAB*(1.+CAB)*(1.+CAB)/CAB
     1 -SCD*(1.+CCD)*(1.+CCD)/CCD)*TH
      D=AA*AE*AI+AD*AH*AC+AB*AF*AG-AC*AE*AG-AB*AD*AI-AF*AH*AA
      THA=(YA*AE*AI+AD*AH*YC+YB*AF*AG-YC*AE*AG-YB*AD*AI-AF*AH*YA)/D
      THD=(AA*YB*AI+YA*AH*AC+AB*YC*AG-AC*YB*AG-AB*YA*AI-YC*AH*AA)/D
      P2=(AA*AE*YC+AD*YB*AC+AB*AF*YA-AC*AE*YA-AB*AD*YC-AF*YB*AA)/D
      THB=(1.+CAB)*TH/CAB-(SAB*G+4.*F)*THA/(SAB*G*CAB)
     1 -2.*F*THD/(SAB*G*CAB)
      THC=(1.+CCD)*TH/CCD-(SCD*G+4.*F)*THD/(SCD*G*CCD)
     1 -2.*F*THA/(SCD*G*CCD)
      PPP=(ABSF(P2-P1))/P1-0.005
      IF(PPP)51,51,50
   50 P1=P2
      GO TO 10
   51 P(J)=P2
      PUNCH 52,TH,P2,THB,THC,THD
   52 FORMAT(//,2X, 7HTHETA= ,F11.8,9H  ALPHA= ,F10.6,7H  THB= ,F6.3,
     17H  THC= ,F6.3,7H  THD= ,F6.3)
      PUNCH 53,RAB,RBC,RCD
   53 FORMAT(41X,5HRAB= ,F6.3,7H  RBC= ,F6.3,7H  RCD= ,F6.3)
      PPPP=P(J)-P(J-1)
      IF(PPPP)70,70,60
   60 IF(J-5)69,63,64
   63 ADDX=2.*ADD
      GO TO 69
   64 IF(J-13)69,65,69
   65 ADDX=3.*ADD
      GO TO 69
   69 J=J+1
      TH=TH+ADDX
      IF(TH-1.05)88,88,70
   88 GO TO 1
   70 CALL EXIT
   62 END
```

Fortran II Symbols

η = F	π = PY
λ = H	θ = TH
γ = G	θ_A = THA
α = P	θ_B = THB
RP = RP	θ_C = THC
REI = REI	θ_D = THD
e = E	

Increment to θ values - ADDX

Initial ADDX value - ADD

S_{AB} = SAB	S_{BC} = SBC	S_{CD} = SCD	F_{BC} = FBC
C_{AB} = CAB	C_{BC} = CBC	C_{CD} = CCD	
ρ_{AB} = RAB	ρ_{BC} = RBC	ρ_{CD} = RCD	
ϕ_{AB} = PAB	ϕ_{BC} = PBC	ϕ_{CD} = PCD	
Sin ϕ_{AB} = SINPAB	Cos ϕ_{AB} = COSPAB		
Sin ϕ_{BC} = SINPBC	Cos ϕ_{BC} = COSPBC	Tan $\frac{\phi_{BC}}{2}$ = TANPBC	
Sin ϕ_{CD} = SINPCD	Cos ϕ_{CD} = COSPCD		
Sinh ϕ_{AB} = SIHPAB	Cosh ϕ_{AB} = COHPAB		
Sinh ϕ_{BC} = SIHPBC	Cosh ϕ_{BC} = COHPBC	Tanh $\frac{\phi_{BC}}{2}$ = TAHPBC	
Sinh ϕ_{CD} = SIHPCD	Cosh ϕ_{CD} = COHPCD		
Sin θ = SINTH	Cos θ = COSTH	Tan θ = TANTH	

e^{ϕ} = EP
$e^{\phi/2}$ = EP2